Algebraic Shifting and f-Vector Theory

Thesis submitted for the degree of
"Doctor of Philosophy"

by
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Submitted to the Senate of the Hebrew University
February 2, 2008

This work was carried out under the supervision of
Prof. Gil Kalai.

Abstract

This manuscript focusses on algebraic shifting and its applications to f-vector theory of simplicial complexes and more general graded posets. It includes attempts to use algebraic shifting for solving the g-conjecture for simplicial spheres, which is considered by many as the main open problem in f-vector theory. While this goal has not been achieved, related results of independent interest were obtained, and are presented here.

The operator algebraic shifting was introduced by Kalai over 20 years ago, with applications mainly in f-vector theory. Since then, connections and applications of this operator to other areas of mathematics, like algebraic topology and combinatorics, have been found by different researchers. See Kalai's recent survey [34]. We try to find (with partial success) relations between algebraic shifting and the following other areas:

- Topological constructions on simplicial complexes.
- Embeddability of simplicial complexes: into spheres and other manifolds.
- f-vector theory for simplicial spheres, and more general complexes.
- f-vector theory for (non-simplicial) graded partially ordered sets.
- Graph minors.

Combinatorially, a (finite) *simplicial complex* is a finite collection of finite sets which is closed under inclusion. This basic object has been subjected to extensive research. Its elements are called *faces*. Its *f-vector* $(f_0, f_1, f_2, ...)$ counts the number of faces according to their dimension, where f_i is the number of its faces of size $i + 1$. f-vector theory tries to characterize the possible f-vectors, by means of numerical relations between the components of the vector, for interesting families of simplicial complexes (and more general objects); for example for simplicial complexes which topologically are spheres.

Algebraic shifting associates with each simplicial complex K a *shifted* simplicial complex, denoted by $\Delta(K)$, which is combinatorially simpler. This is an invariant which on the one hand preserves important invariants of K, like its f-vector and Betti numbers, while on the other hand loses other invariants, like the topological, and even homotopical, type of K. A general problem is to understand which invariants of K can be read off from the faces of $\Delta(K)$, and how. There are two different variations of this operator: one is based on the exterior algebra, the other - on the symmetric algebra; both

were introduced by Kalai. Many statements are true, or conjectured to be true, for both variations. (Definitions appear in the next chapter.)

The main open problem in f-vector theory is to characterize the f-vectors of *simplicial spheres* (i.e. simplicial complexes which are homeomorphic to spheres). The widely believed conjecture for the last 25 years, known as the g-conjecture, is that the characterization for simplicial convex polytopes, proved by Stanley (necessity)[68], and by Billera and Lee (sufficiency)[6], holds also for the wider class of simplicial spheres, and even for all homology spheres, i.e. Gorenstein* complexes. Its open part is to show that the g-vector, which is determined by the f-vector, is an M-sequence for these simplicial complexes.

The algebraic properties of face rings hard-Lefschetz and weak-Lefschetz translate into certain properties of the symmetrically shifted complex. Having any of these properties in the face ring of simplicial spheres would imply the g-conjecture. A conjecture by Kalai and by Sarkaria, stating which faces are never in $\Delta(K)$ if K can be embedded in a sphere, would also imply the g-conjecture for simplicial spheres [34]. The well known lower bound and upper bound theorems for f-vectors of simplicial spheres have algebraic shifting conjectured refinements, which immediately imply these theorems. Details appear in Chapters 4 and 5. Partial results on these conjectures include:

- The known lower bound inequalities for simplicial spheres are shown to hold for the larger class of doubly Cohen-Macaulay (2-CM) complexes, by using algebraic shifting / rigidity theory for graphs and Fogelsanger's theory of *minimal cycle complexes* [24]. Moreover, the initial part (g_0, g_1, g_2) of the g-vector of a 2-CM complex (of dimension ≥ 3) is shown to be an M-sequence. This supports the conjecture by Björner and Swartz that the entire g-vector of a 2-CM complex is an M-sequence [72]. See Section 3.5.

- The non-negativity of the g-vector, which translates to the generalized lower bound inequalities on the f-vector, is proved for a special class of simplicial spheres, by using special edge contractions. These contractions are well behaved with respect to properties of the face rings of those simplicial complexes. To obtain this result, we first answer affirmatively a problem asked by Dey et. al. [19] concerning topology-preserving edge contractions in PL-manifolds. See Sections 5.4 and 4.6.

- The hard-Lefschetz property is preserved under the combinatorial operations of join, Stellar subdivisions and connected sum of spheres; thus supporting the g-conjecture. See Sections 4.2, 4.6 and 4.4.

The (generic) rigidity property for a graph mapped into a Euclidean space of fixed dimension is equivalent to the existence of a certain edge in the symmetric algebraic shifting of the graph. Similarly, the dimension of the space of stresses in a generic embedding equals the number of edges of a certain type in its symmetric shifting. This follows from a work by Lee [39]. Analogues for exterior shifting involves Kalai's notion of hyperconnectivity [31]. We use these connections, together with graph minors, to conclude the following:

- Shifting can tell minors: for every $2 \leq r \leq 6$ and every graph G, if $\{r-1, r\} \in \Delta(G)$ then G has a K_r minor. As a corollary, obstructions to embeddability are obtained. See Section 3.6.

- Higher dimensional analogues: we generalized the notion of *minors* in graphs to the class of arbitrary simplicial complexes, and proved that it 'respects' the Van-Kampen obstruction in equivariant cohomology. This suggests a new approach for proving the Kalai-Sarkaria conjecture (and hence the g-conjecture). Details appear in Chapter 5.

Algebraic shifting of more general graded partially ordered sets than simplicial complexes may be used to prove some of their properties by looking at the shifted object. For example, an open problem is to show that the (toric) g-vector of a general polytope is an M-sequence. The above approach may be useful in proving it. Recently Karu has proved that this g-vector is non-negative, by algebraic means. We obtained the following progress in this direction:

- We defined an algebraic shifting operator for geometric meet semi-lattices (simplicial complexes are an important example of these objects), by constructing face rings for these objects. This generalizes the construction for simplicial complexes. As an application, we reprove the fact that their f-vector satisfies the Kruskal-Katona inequalities, proved by Wegner [77]. See Section 6.2, and the rest of Chapter 6 for further results in the same spirit.

Apart from applications, algebraic shifting became an interesting research object by its own right, as indicated by numerous recent papers done by multiple researchers.

- We describe the behavior of algebraic shifting with respect to some basic constructions on simplicial complexes, like union, cone and more generally, join. For this, a 'homological' point of view on algebraic shifting was developed. Interestingly, a multiplicative formula obtained

for exterior shifting of joins, fails for symmetric shifting. The main applications are as follows; see Chapter 2 for details.

- Proving Kalai's conjecture [34] that if K and L are disjoint simplicial complexes, then $\Delta(K \cup L) = \Delta(\Delta(K) \cup \Delta(L))$.
- Disproving Kalai's conjecture for joins [34], by providing examples where $\Delta(K * L)$ is not equal to $\Delta(\Delta(K) * \Delta(L))$.
- A new proof for Kalai's formula for exterior shifting of a cone $\Delta^e(C(K)) = C(\Delta^e(K))$.

To summarize, the operator algebraic shifting is a powerful tool for proving claims in f-vector theory and has relations to the above mentioned areas in mathematics. A better understanding of this operator and these relations may be used to prove well known open problems like the ones indicated here, and is also interesting by its own right. Partial success in achieving this goal is presented in this manuscript. However, it seems that the potential of this tool has not yet been exhausted.

Most of the results presented here can be found in papers (except for those in Chapter 4), as follows: most of Chapters 1 and 2 in [54]; of Chapter 3 in [52] and [53]; most of Chapter 5 in [51]; of Chapter 6 in [55]. Each chapter ends with related open problems and conjectures.

I hope you will enjoy the reading.

Acknowledgements

I would like to express my profound thanks to my advisor prof. Gil Kalai, for numerous helpful and inspiring mathematical discussions, as well as for his encouragement, support and care during the years of work on this thesis.

During those years I had fruitful discussions with various mathematicians, which reflect directly in this work. Discussions with Eric Babson about the content of Chapters 4 and 5 led to the results about join and Stellar subdivisions in Chapter 4; with Yhonatan Iron to Proposition 4.3.1; with Carsten Thomassen to the proofs in Subsection 3.6.2; as well as helpful discussions with Ed Swartz about the content of Chapter 4.

With many other mathematicians as well I had interesting and inspiring discussions concerning this work, including Nir Avni, Uri Bader, Anders Björner, Nati Linial, Isabella Novik, Vic Reiner, Uli Wagner and Volkmar Welker. I am thankful to all of them.

During 2005 I attended the program on algebraic combinatorics held in Institut Mittag-Leffler. This period was very fruitful and pleasant for me. I deeply thank the organizers Anders Björner and Richard Stanley for the invitation, and my host Svante Linusson at the Swedish node of the "Algebraic Combinatorics in Europe" program, which kindly supported my stay there.

In the last two years I have been supported by a Charles Clore Foundation Fellowship for Ph.D. students. This fellowship allowed me to participate in conferences, to devote more time to research and to study tranquilly. I thank Dame Vivien Duffield and the Clore Foundation for this support.

I also wish to thank the people of the math department at the Hebrew University for a pleasant and supportive atmosphere to work in.

Finally, I take this opportunity to thank my family and friends, Anna and Marni.

Contents

Chapter 1

Basic Definitions and Concepts

1.1 Basics of simplicial complexes

Let $[n] = \{1, 2, ..., n\}$, and $\binom{[n]}{k}$ denote the subsets of $[n]$ of size k. A collection K of subsets of $[n]$ is called a (finite abstract) *simplicial complex* if it is closed under inclusion, i.e. $S \subseteq T \in K$ implies $S \in K$. Note that if K is not empty (which we will assume from now on) then $\emptyset \in K$. The i-th skeleton of K is $K_i = \{S \in K : |S| = i + 1\} = K \cap \binom{[n]}{i+1}$. The elements of K are called *faces*; those in K_i have *dimension* i. The 0-dimensional faces are called *vertices*, the 1-dimensional faces are called *edges* and the maximal faces with respect to inclusion are called *facets*. If all the facets have the same dimension, K is *pure*. The *f-vector* (face vector) of K is $f(K) = (f_{-1}, f_0, f_1, ...)$ where $f_i = |K_i|$. The *dimension of* K is $\dim(K) := \max\{i : f_i(K) \neq 0\}$; e.g. a 1-dimensional simplicial complex is a simple graph. The *f-polynomial* of K is $f(K, t) = \sum_{i \geq 0} f_{i-1} t^i$.

The *link* of a face $S \in K$ is $\mathrm{lk(S, K)} = \{\mathrm{T \in K : T \cap S = \emptyset,\ T \cup S \in K}\}$. Note that $\mathrm{lk(S, K)}$ is also a simplicial complex, and that $\mathrm{lk(\emptyset, K) = K}$. The (open) *star* of $S \in K$ is $\mathrm{st}(S, K) = \{T \in K : S \subseteq T\}$, which is not a simplicial complex; the *closed star* of S in K is $\overline{\mathrm{st}}(S, K) = \{T \in K : T \cup S \in K\}$, which is a simplicial complex. The *anti star* of $S \in K$ is $\mathrm{ast}(S, K) = \{T \in K : S \cap T = \emptyset\}$, which is a simplicial complex. The join of two simplicial complexes K, L with disjoint sets of vertices is the simplicial complex $K * L = \{S \cup T : S \in K, T \in L\}$. Note that $f(K * L, t) = f(K, t) f(L, t)$.

A *simplex* in $\mathbb{R}^N$ is the convex hull of some affinely independent points in $\mathbb{R}^N$. Its intersection with a supporting hyperplane is a *face* of it, as well as the empty face. The 0-dimensional simplices are called vertices. A (finite) *geometric simplicial complex* L in $\mathbb{R}^N$ is a finite collection of simplices in $\mathbb{R}^N$ such that:

(a) Every face of a simplex in L is in L.
(b) The intersection of any two simplices in L is a face of each of them.
We endow the union of simplices in L with the induces topology as a subspace of the Euclidian space $\mathbb{R}^N$, and call it the *topology of* L. A *geometric realization* of an abstract simplicial complex K is a geometric simplicial complex L in some $\mathbb{R}^N$ such that L is combinatorially isomorphic to K, i.e. as posets w.r.t. inclusion. Any two geometric realizations of the same simplicial complex, K, are (piecewise linearly) homeomorphic, and we denote this topological space by $||K||$. We refer to topological properties of $||K||$ as properties of K; e.g. K_5, the complete graph on 5 vertices, is not embeddable in $\mathbb{R}^2$. We say that a simplicial complex K is a *triangulation* of a topological space X if $||K||$ is homeomorphic to X.

Let $\tilde{H}_i(K;k)$ denote the reduced i-th (simplicial) homology group of K with field k coefficients. $\beta_i = \beta_i(K;k) = \dim_k(\tilde{H}_i(K,k))$ is the i-th *Betti number* of K with k coefficients, and $\beta(K;k) = (\beta_0, \beta_1, ...)$ is its *Betti vector*.

Let $<$ denote the usual order on the natural numbers. A simplicial complex K with vertices $[n]$ is *shifted* if for every $i < j$, $j \in S \in K$, also $(S \setminus \{j\}) \cup \{i\} \in K$. Let $<_P$ be the product partial order on equal sized ordered subsets of $\mathbb{N}$. That is, for $S = \{s_1 < ... < s_i\}$ and $T = \{t_1 < ... < t_i\}$ $S \leq_P T$ iff $s_j \leq t_j$ for every $1 \leq j \leq i$. Then K is shifted iff $S <_P T \in K$ implies $S \in K$. It is easy to see that every simplicial complex has a shifted simplicial complex with the same f-vector: for some $1 \leq i < j \leq n$ apply $S \mapsto (S \setminus \{j\}) \cup \{i\}$ for all $j \in S \in K, i \notin S$ to obtain K', which is also a simplicial complex, with the same f-vector. Repeat this process as long as possible, to obtain a shifted simplicial complex $\Delta^c(K)$. Note that $\Delta^c(K)$ depends on the order of choices of pairs $i < j$. The operation $K \mapsto \Delta^c(K)$ is called *combinatorial shifting*, introduced by Erdš Ko and Rado [22].

Note that a shifted complex K is homotopy equivalent to a wedge of spheres, where the number of i-dimensional spheres in this wedge is $|\{S \in K_i : S \cup \{1\} \notin K\}|$. In particular, its Betti numbers are easily read off from its combinatorics: $\beta_i(K;k) = |\{S \in K_i : S \cup \{1\} \notin K\}|$ for every field k.

For further details about simplicial complexes, and about simplicial homology, we refer to Munkres' book [48].

Another useful way to encode the information in the f-vector $f(K)$ is by the *h-vector*. Let K be $(d-1)$-dimensional, and define

$$\sum_{0 \leq i \leq d} h_i(K) x^{d-i} = \sum_{0 \leq i \leq d} f_{i-1}(K)(x-1)^{d-i}.$$

Equivalently, $h_k = \sum_{0 \leq i \leq k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}$ and $f_{k-1} = \sum_{0 \leq i \leq k} \binom{d-i}{k-i} h_i$. The *h-polynomial* of K is $h(K,t) = \sum_{i \geq 0} h_i(K) t^i$, hence $(1+t)^d h(K, \frac{1}{t+1}) =$

$t^d f(K, \frac{1}{t})$. If $||K||$ is a sphere, then $h(K)$ is symmetric, i.e. $h_i(K) = h_{d-i}(K)$ for every $0 \leq i \leq d$. Equivalently, $h(K,t) = t^d h(K, \frac{1}{t})$. These relations are known as Dehn-Sommerville equations. In this case $f(K)$ can be recovered from the *g-vector*: $g_0(K) := h_0(K) = 1$, $g_i(K) := h_i(K) - h_{i-1}(K)$ for $1 \leq i \leq \lfloor d/2 \rfloor$. $g(K) := (g_0(K), ..., g_{\lfloor d/2 \rfloor}(K))$.

For more information about the importance of the h- and g-vectors, we refer to Stanley's book [68], and to Chapter 4 below.

1.2 Basics of algebraic shifting

We give the definition of exterior and symmetric algebraic shifting and state some of there basic properties. We develop a 'dual' point of view which leads to equivalent definitions, and use these 'dual' definitions to cut the faces of the shifted complex into 'intervals' which play a crucial role in the proofs of the results of Chapter 2.

1.2.1 Exterior shifting

Via the exterior algebra

Let $\mathbb{F}$ be a field and let k be a field extension of $\mathbb{F}$ of transcendental degree $\geq n^2$ (e.g. $\mathbb{F} = \mathbb{Q}$ and $k = \mathbb{R}$, or $\mathbb{F} = \mathbb{Z}_2$ and $k = \mathbb{Z}_2(x_{ij})_{1 \leq i,j \leq n}$ where x_{ij} are intermediates). Let V be an n-dimensional vector space over k with basis $\{e_1, \ldots, e_n\}$. Let $\bigwedge V$ be the graded exterior algebra over V. Denote $e_S = e_{s_1} \wedge \cdots \wedge e_{s_j}$ where $S = \{s_1 < \cdots < s_j\}$. Then $\{e_S : S \in \binom{[n]}{j}\}$ is a basis for $\bigwedge^j V$. Note that as K is a simplicial complex, the ideal $(e_S : S \notin K)$ of $\bigwedge V$ and the vector subspace $\mathrm{span}\{e_S : S \notin K\}$ of $\bigwedge V$ consist of the same set of elements in $\bigwedge V$. Define the exterior algebra of K by

$$\bigwedge K = (\bigwedge V)/(e_S : S \notin K).$$

Let $\{f_1, \ldots, f_n\}$ be a basis of V, generic over $\mathbb{F}$ with respect to $\{e_1, \ldots, e_n\}$, which means that the entries of the corresponding transition matrix A ($e_i A = f_i$ for all i) are algebraically independent over $\mathbb{F}$. Let $\tilde{f}_S$ be the image of $f_S \in \bigwedge V$ in $\bigwedge(K)$. Let $<_L$ be the lexicographic order on equal sized subsets of $\mathbb{N}$, i.e. $S <_L T$ iff $\min(S \triangle T) \in S$. Define

$$\Delta^e(K) = \Delta^e_A(K) = \{S : \tilde{f}_S \notin \mathrm{span}\{\tilde{f}_{S'} : S' <_L S\}\}$$

to be the shifted complex, introduced by Kalai [30]. The construction is canonical, i.e. it is independent of the choice of the generic matrix A, and

for a permutation $\pi : [n] \to [n]$ the induced simplicial complex $\pi(K)$ satisfies $\Delta^e(\pi(K)) = \Delta^e(K)$. It results in a shifted simplicial complex, having the same face vector and Betti vector as K's [8]. Some more of its basic properties are detailed in subsection 1.2.3.

Via dual setting

Fixing the basis $\{e_1, \ldots, e_n\}$ of V induces the basis $\{e_S : S \subseteq [n]\}$ of $\bigwedge V$ as a k-vector space, which in turn induces the dual basis $\{e^*_T : T \subseteq [n]\}$ of $(\bigwedge V)^*$ by defining $e^*_T(e_S) = \delta_{T,S}$ and extending bilinearly. $(\bigwedge V)^*$ stands for the space of k-linear functionals on $\bigwedge V$. For $f, g \in \bigwedge V$ $< f, g >$ will denote $f^*(g)$. Define the so called left interior product of g on f [31], where $g, f \in \wedge V$, denoted $g\lfloor f$, by the requirement that for all $h \in \bigwedge V$

$$< h, g\lfloor f > = < h \wedge g, f > .$$

$g\lfloor\cdot$ is the adjoint operator of $\cdot \wedge g$ w.r.t. the bilinear form $< \cdot, \cdot >$ on $\bigwedge V$. Thus, $g\lfloor f$ is a bilinear function, satisfying

$$e_T\lfloor e_S = \{^{(-1)^{a(T,S)} e_{S \setminus T} \ if \ T \subseteq S}_{0 \ otherwise} \tag{1.1}$$

where $a(T, S) = |\{(s, t) \in S \times T : s \notin T, t < s\}|$. This implies in particular that for a monomial g (i.e. g is a wedge product of elements of degree 1) $g\lfloor$ is a boundary operation on $\bigwedge V$, and in particular on $\mathrm{span_k}\{e_S : S \in K\}$ [31]. Let $\bigwedge^j(K) = \mathrm{span_k}\{e_S : S \in K_{j-1}\}$ and $\bigwedge(K) = \mathrm{span_k}\{e_S : S \in K\}$. We denote:

$$\mathrm{Ker}_j\, f_R\lfloor(K) = \mathrm{Ker}_j\, f_R\lfloor = \mathrm{Ker}(f_R\lfloor : \bigwedge^{j+1}(K) \to \bigwedge^{j+1-|R|}(K)).$$

Note that the definition of $\bigwedge(K)$ makes sense more generally when $K_0 \subseteq [n]$ (and not merely when $K_0 = [n]$), and still $f_R\lfloor$ operates on the subspace $\bigwedge(K)$ of $\bigwedge V$ for every $R \subseteq [n]$. (Recall that $f_i = \alpha_{i1}e_1 + ... + \alpha_{in}e_n$ where $A = (\alpha_{ij})_{1 \le i,j \le n}$ is a generic matrix.) Define $f^0_i = \sum_{j \in K_0} \alpha_{ij}e_j$, and $f^0_S = f^0_{s_1} \wedge \cdots \wedge f^0_{s_j}$ where $S = \{s_1 < \cdots < s_j\}$. By equation (1.1), the following equality of operators on $\bigwedge(K)$ holds:

$$\forall S \subseteq [n] \quad f_S\lfloor = f^0_S\lfloor. \tag{1.2}$$

Now we give some equivalent descriptions of the exterior shifting operator, using the kernels defined above. This approach will be used in Chapter 2. The following generalizes a result for graphs [31] (the proof is similar):

Proposition 1.2.1 *Let* $R \subseteq [n]$, $|R| < j+1$. *Then:*

$$\mathrm{Ker}_j\, f_R\lfloor = \bigcap_{i:[n]\ni i\notin R} \mathrm{Ker}_j\, f_{R\cup i}\lfloor.$$

Proof: Recall that $h\lfloor(g\lfloor f) = (h \wedge g)\lfloor f$. Thus, if $f_R\lfloor m = 0$ then

$$f_{R\cup i}\lfloor m = \pm f_i\lfloor(f_R\lfloor m) = f_i\lfloor 0 = 0.$$

Now suppose $m \in \bigwedge^{j+1}(K) \setminus \mathrm{Ker}_j\, f_R\lfloor$. The set $\{f_Q^* : Q \subseteq [n], |Q| = j + 1 - |R|\}$ forms a basis of $(\bigwedge^{j+1-|R|} V)^*$, so there is some $f_{R'}$, $R' \subseteq [n], |R'| = j + 1 - |R|$, (note that $R' \neq \emptyset$) such that

$$< f_{R'} \wedge f_R, m > = < f_{R'}, f_R\lfloor m > \neq 0.$$

We get that for $i_0 \in R'$: $i_0 \notin R$ and $< f_{R'\setminus i_0}, f_{R\cup i_0}\lfloor m > \neq 0$. Thus $m \notin \mathrm{Ker}_j\, f_{R\cup i_0}\lfloor$ which completes the proof.$\square$

In the next proposition we determine the shifting of a simplicial complex by looking at the intersection of kernels of boundary operations (actually only at their dimensions): Let S be a subset of $[n]$ of size s. For $R \subseteq [n]$,$|R| = s$, we look at $f_R\lfloor: \bigwedge^s(K) \to \bigwedge^{s-|R|}(K) = k$.

Proposition 1.2.2 *Let* $K_0, S \subseteq [n], |K_0| = k, |S| = s$. *The following quantities are equal:*

$$\dim \bigcap_{R<_L S,|R|=s,R\subseteq[n]} \mathrm{Ker}_{s-1}\, f_R\lfloor, \tag{1.3}$$

$$\dim \bigcap_{R<_L S,|R|=s,R\subseteq[k]} \mathrm{Ker}_{s-1}\, f_R^0\lfloor, \tag{1.4}$$

$$|\{T \in \Delta^e(K) : |T| = s, S \leq_L T\}|. \tag{1.5}$$

In particular, $S \in \Delta^e(K)$ *iff*

$$\dim \bigcap_{R<_L S,|R|=s,R\subseteq[n]} \mathrm{Ker}_{s-1}\, f_R\lfloor > \dim \bigcap_{R\leq_L S,|R|=s,R\subseteq[n]} \mathrm{Ker}_{s-1}\, f_R\lfloor$$

(equivalently, $S \in \Delta^e(K)$ *iff* $\bigcap_{R<_L S,|R|=s,R\subseteq[n]} \mathrm{Ker}_{s-1}\, f_R\lfloor \nsubseteq \mathrm{Ker}_{s-1}\, f_S\lfloor$*).*

Proof: First we show that (1.3) equals (1.4). For every $T \subseteq [n]$, $T <_L S$, decompose $T = T_1 \cup T_2$, where $T_1 \subseteq [k], T_2 \cap [k] = \emptyset$. For each T_3 satisfying $T_3 \subseteq [k], T_3 \supseteq T_1, |T_3| = |T|$, we have $T_3 \leq_L T$. Each f_t^0, where $t \in T_2$, is a

linear combination of the f_i^0's, $1 \leq i \leq k$, so f_T^0 is a linear combination of such $f_{T_3}^0$'s. Thus, for every $j \geq s-1$,

$$\bigcap_{R \leq_L T, R \subseteq [k]} \mathrm{Ker}_j\, f_R^0 \lfloor \subseteq \mathrm{Ker}_j\, f_T^0 \lfloor,$$

and hence

$$\bigcap_{R <_L S, R \subseteq [k]} \mathrm{Ker}_j\, f_R^0 \lfloor = \bigcap_{R <_L S, R \subseteq [n]} \mathrm{Ker}_j\, f_R^0 \lfloor.$$

Combining with (1.2) the desired equality follows.

Next we show that (1.5) equals (1.4). Let $m \in \bigwedge^s(K)$ and $R \subseteq [n], |R| = s$. Let us express m and f_R in the basis $\{e_S : S \subseteq [n]\}$:

$$m = \sum_{T \in K, |T|=s} \gamma_T e_T, \;\; f_R = \sum_{S' \subseteq [n], |S'|=s} A_{RS'} e_{S'}$$

where $A_{RS'}$ is the minor of A (transition matrix) with respect to the rows R and columns S', and where γ_T is a scalar in $\mathbb{R}$.

By bilinearity we get

$$f_R^0 \lfloor m = f_R \lfloor m = \sum_{T \in K, |T|=s} \gamma_T A_{RT}.$$

Thus (1.4) equals the dimension of the solution space of the system $B_S x = 0$, where B_S is the matrix (A_{RT}), where $R <_L S$, $R \subseteq [k]$, $|R| = s$ and $T \in K, |T| = s$. But, since the row indices of B_S are an initial set with respect to the lexicographic order, the intersection of $\Delta^e(K)$ with this set of indices determines a basis of the row space of B_S. Thus, $\mathrm{rank}(B_S) = |\{R \in \Delta^e(K) : |R| = s, R <_L S\}|$. But K and $\Delta^e(K)$ have the same f-vector, so we get:

$$\dim \bigcap_{R <_L S, R \subseteq [k], |R|=s} \mathrm{Ker}_{s-1}\, f_R^0 \lfloor = f_{s-1}(K) - \mathrm{rank}(B_S) =$$

$$= |\{T \in \Delta^e(K) : |T| = s, S \leq_L T\}|$$

as desired.□

Dividing $\Delta^e(K)$ into intervals

For each $j > 0$ and $S \subseteq [n]$, $|S| \geq j$ we define $\mathrm{init}_j(S)$ to be the set of lexicographically least j elements in S, and for every $i > 0$ define

$$I_S^i = I_S^i(n) = \{T : T \subseteq [n], |T| = |S| + i, \mathrm{init}_{|S|}(T) = S\}.$$

Let $S_{(i)}^{(m)} = S_{(i)}^{(m)}(n) = \min_{<_L} I_S^i(n)$ and $S_{(i)}^{(M)} = S_{(i)}^{(M)}(n) = \max_{<_L} I_S^i(n)$. In the sequel, all the sets of numbers we consider are subsets of $[n]$. In order to simplify notation, we will often omit noting that. We get the following information about the partition of the faces in the shifted complex into 'intervals':

Proposition 1.2.3 *Let $K_0 \subseteq [n]$, $S \subseteq [n]$, $i > 0$. Then:*

$$|I_S^i \cap \Delta^e(K)| = \dim \bigcap_{R <_L S} \mathrm{Ker}_{|S|+i-1} f_R \lfloor (K) - \dim \bigcap_{R \leq_L S} \mathrm{Ker}_{|S|+i-1} f_R \lfloor (K).$$

Proof: By Proposition 1.2.1,

$$\dim \bigcap_{R <_L S} \mathrm{Ker}_{|S|+i-1} f_R \lfloor = \dim \bigcap_{R <_L S} \bigcap_{j \notin R, j \in [n]} \mathrm{Ker}_{|S|+i-1} f_{R \cup j} \lfloor = \ldots =$$

$$\dim \bigcap_{R <_L S} \bigcap_{T : T \cap R = \emptyset, |T| = i} \mathrm{Ker}_{|S|+i-1} f_{R \cup T} \lfloor = \dim \bigcap_{R <_L S_{(i)}^{(m)}} \mathrm{Ker}_{|S_{(i)}^{(m)}|-1} f_R \lfloor .$$

To see that the last equation is true, one needs to check that $\{R \cup T : T \cap R = \emptyset, |T| = i, R <_L S\} = \{Q : Q <_L S_{(i)}^{(m)}\}$. By Proposition 1.2.2,

$$\dim \bigcap_{R <_L S_{(i)}^{(m)}} \mathrm{Ker}_{|S_{(i)}^{(m)}|-1} f_R \lfloor = |\{Q \in \Delta^e(K) : |Q| = |S| + i, S_{(i)}^{(m)} \leq_L Q\}|.$$

Similarly,

$$\dim \bigcap_{R \leq_L S} \mathrm{Ker}_{|S|+i-1} f_R \lfloor (K) = |\{F \in \Delta^e(K) : |F| = |S| + i, S_{(i)}^{(M)} <_L F\}|.$$

Here one checks that $\{R \cup T : T \cap R = \emptyset, |T| = i, R \leq_L S\} = \{F : F \leq_L S_{(i)}^{(M)}\}$. Thus, the proof of the proposition is completed. $\square$

Note that on I_S^1 the lexicographic order and the partial order $<_P$ coincide, since all sets in I_S^1 have the same $|S|$ least elements. As $\Delta^e(K)$ is shifted, $I_S^1 \cap \Delta^e(K)$ is an initial set of I_S^1 with respect to $<_L$. Denote for short

$$D(S) = D_K(S) = |I^1_{\mathrm{init}_{|S|-1}(S)}(n) \cap \Delta^e(K)|.$$

$D_K(S)$ is indeed independent of the particular n we choose, as long as $K_0 \subseteq [n]$. We observe that:

Proposition 1.2.4 *Let K_0 and $S = \{s_1 < \cdots < s_j < s_{j+1}\}$ be subsets of $[n]$. Then: $S \in \Delta^e(K) \Leftrightarrow s_{j+1} - s_j \leq D(S)$.* $\square$

Another easy preparatory lemma is the following:

Proposition 1.2.5 *Let $K_0, S \subseteq [n]$. Then: $D_{\Delta^e(K)}(S) = D_K(S)$.*

Proof: It follows from the fact that $\Delta^e \circ \Delta^e = \Delta^e$ (Kalai [33], or later on here in Corollary 2.2.7). $\square$

1.2.2 Symmetric shifting

Via the symmetric algebra

For symmetric shifting, let us look on the face ring (Stanley-Reisner ring) of K $k[K] = k[x_1, .., x_n]/I_K$ where I_K is the homogenous ideal generated by the monomials whose support is not in K, $\{\prod_{i\in S} x_i : \ S \notin K\}$. $k[K]$ is graded by degree. Let $\mathbb{F} \subseteq k$ be fields as before and let $y_1, \ldots, y_n$ be generic linear combinations of $x_1, \ldots, x_n$ w.r.t. $\mathbb{F}$. We choose a basis for each graded component of $k[K]$, up to degree $\dim(K)+1$, from the canonic projection of the monomials in the y_i's on $k[K]$, in the greedy way:

$$\mathrm{GIN}(K) = \{m : \tilde{m} \notin \mathrm{span_k}\{\tilde{m'} : \deg(m') = \deg(m), m' <_L m\}\}$$

where $\prod y_i^{a_i} <_L \prod y_i^{b_i}$ iff for $j = \min\{i : a_i \neq b_i\}$ $a_j > b_j$). The combinatorial information in $\mathrm{GIN}(K)$ is redundant: if $m \in \mathrm{GIN}(K)$ is of degree $i \leq \dim(K)$ then $y_1 m, .., y_i m$ are also in $\mathrm{GIN}(K)$. Thus, $\mathrm{GIN}(K)$ can be reconstructed from its monomials of the form $m = y_{i_1} \cdot y_{i_2} \cdot .. \cdot y_{i_r}$ where $r \leq i_1 \leq i_2 \leq .. \leq i_r$, $r \leq \dim(K) + 1$. Denote this set by gin(K), and define $S(m) = \{i_1 - r + 1, i_2 - r + 2, .., i_r\}$ for such m. The collection of sets

$$\Delta^s(K) = \cup\{S(m) : m \in \mathrm{gin(K)}\}$$

carries the same combinatorial information as $\mathrm{GIN}(K)$. $\Delta^s(K)$ is a simplicial complex. Again, the construction is canonic, in the same sense as for exterior shifting. If k has characteristic zero then $\Delta^s(K)$ is shifted [33]. Further basic properties of $\Delta^s(K)$ are detailed in subsection 1.2.3.

Via dual setting

Denote monomials in the graded polynomial ring $R = k[x_1, .., x_n] = k \oplus R_1 \oplus R_2 \oplus ...$ by $x^a = \prod_{1\leq i\leq n} x_i^{a_i}$ where $a_i \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$, and define a bilinear form on R by $< x^a, x^b >= \delta_{a,b}$. For a subspace A of R denote its orthogonal subspace by $A^\perp$. Every element $m \in R$ defines a map on R by multiplication $m : \ r \mapsto mr$, and denote its adjoint map by m^*. This induces a bilinear map on R, $m \cdot u = m^*(u)$ which satisfies

$$x^a \cdot x^b = \begin{cases} x^{b-a} & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

where $a \leq b$ means that componentwise $a_i \leq b_i$. Thus, for a simplicial complex K with $K_0 \subseteq [n]$, the restriction $m^*_{|I_K^\perp}$ is into $I_K^\perp$, as a basis for this subspace is $\{x^a : \mathrm{supp}(a) \in K\}$ where $\mathrm{supp}(a) = \{i : a_i > 0\}$. Denote the degree i part of $I_K^\perp$ by $(I_K^\perp)_i$, and the degree of an element m by $\deg(m)$. For a homogenous element m, let $\mathrm{Ker}_j(m^*) = \mathrm{Ker}(m^* : \ (I_K^\perp)_{j+1} \to (I_K^\perp)_{j+1-\deg(m)})$.

Let $Y = \{y_1, ..., y_n\}$ be a generic basis for R_1 w.r.t. the basis $X = \{x_1, ..., x_n\}$.

Proposition 1.2.6 *Let m be a monomial in the basis Y (i.e. $m = \prod_i y_i^{a_i}$), and $j \geq \deg(m)$. Then $\operatorname{Ker}_j m^* = \bigcap_{i\in[n]} \operatorname{Ker}_j (my_i)^*$.*

Proof: By the associativity and commutativity of multiplication in R, when passing to adjoint maps one obtains $(my_i)^* = y_i^* \circ m^*$, and hence $\operatorname{Ker}_j m^* \subseteq \bigcap_{i\in[n]} \operatorname{Ker}_j (my_i)^*$. Conversely, if $x \in (I_K^\perp)_{j+1} \setminus \operatorname{Ker}_j m^*$, there exists a monomial y in the basis Y such that $< y, m^*(x) > \neq 0$ (as such monomials y span R over k). Write $y = y_{i_0} y'$ for suitable i_0 (this is possible as $j \geq \deg(m)$). Then $0 \neq < y, m^*(x) > = < y', y_{i_0}^*(m^*(x)) >$, in particular $x \notin \operatorname{Ker}(my_{i_0})^*$ $\square$
Convention: From now on when writing the relation of monomials $z <_L y$ it will mean that we assume $\deg(z) = \deg(y)$ even if we do not say so explicitly.

Proposition 1.2.7 *Let y, z be monomials in the basis Y. Then*

$$|\{z \in \mathrm{GIN}(K) : \deg(z) = \deg(y), y \leq_L z\}| = \dim \bigcap_{z <_L y} \operatorname{Ker}_{\deg(y)-1} z^*.$$

In particular,

$$y \in \mathrm{GIN}(K) \iff \bigcap_{z <_L y} \operatorname{Ker}_{\deg(y)-1} z^* \nsubseteq \operatorname{Ker}_{\deg(y)-1} y^*.$$

Proof: First note that the intersection on RHS does not change when replacing the z's with their orthogonal projection on $I_K^\perp$ (as for $i \in [n] \setminus K_0$, $x_i^*(I_K^\perp) = 0$), thus we may assume $[n] = K_0$.

Consider the $|\{z : z <_L y, \deg(z) = \deg(y)\}| \times \dim(I_K^\perp)_{\deg(y)}$ matrix A with $A_{z,x^a} = < z, x^a >$. Then $RHS = \dim_k(\operatorname{Ker}(A)) = \dim_k((I_K^\perp)_{\deg(y)}) - \dim_k(\operatorname{Im}(A)) = LHS$. $\square$

Dividing GIN(K) into intervals

For a monomial y^a and $i \leq \deg(y^a)$ let $\operatorname{init}_i(y^a)$ be the lexicographically least monomial of degree i which divides y^a. For every $i > 0$ define the following subsets of monomials with variables in Y:

$$J_y^i = J_y^i(n) = \{m \in \mathbb{Z}_+^Y : y|m, \deg(m) = \deg(y) + i, \operatorname{init}_{\deg(y)}(m) = y\}.$$

Let $y_{(i)}^{(m)} = y_{(i)}^{(m)}(n) = \min_{<_L} J_y^i(n)$ and $y_{(i)}^{(M)} = y_{(i)}^{(M)}(n) = \max_{<_L} J_y^i(n)$.

Proposition 1.2.8 *Let $K_0 \subseteq [n]$, $S \subseteq [n]$, $i > 0$. Then:*

$$|J_y^i \cap \mathrm{GIN}(K)| = \dim \bigcap_{z <_L y} \mathrm{Ker}_{\deg(y)+i-1}\, z^* - \dim \bigcap_{z \leq_L y} \mathrm{Ker}_{\deg(y)+i-1}\, z^*.$$

Proof: By Proposition 1.2.6,

$$\dim \bigcap_{z <_L y} \mathrm{Ker}_{\deg(y)+i-1}\, z^* = \dim \bigcap_{z <_L y} \bigcap_{j \in [n]} \mathrm{Ker}_{\deg(y)+i-1} (zy_j)^* = \ldots =$$

$$\dim \bigcap_{z <_L y} \bigcap_{t \in (\mathbb{Z}_+^Y)_i} \mathrm{Ker}_{\deg(y)+i-1} (zt)^* = \dim \bigcap_{z <_L y_{(i)}^{(m)}} \mathrm{Ker}_{\deg(y_{(i)}^{(m)})-1}\, z^*$$

which, by Proposition 1.2.7, equals

$$|\{m \in \mathrm{GIN}(K) : \deg(m) = \deg(y) + i,\ y_{(i)}^{(m)} \leq_L m\}|.$$

Similarly,

$$\dim \bigcap_{z \leq_L y} \mathrm{Ker}_{\deg(y)+i-1}\, z^* = \dim \bigcap_{z \leq_L y_{(i)}^{(M)}} \mathrm{Ker}_{\deg(y_{(i)}^{(M)})-1}\, z^* =$$

$$|\{m \in \mathrm{GIN}(K) : \deg(m) = \deg(y) + i,\ y_{(i)}^{(M)} <_L m\}|.$$

$\square$

1.2.3 Basic properties

Both exterior and symmetric shifting share the following basic properties; see [34] for more details and references to the original proofs. We denote both shifting operators by $K \mapsto \Delta(K)$.

Theorem 1.2.9 *(Kalai) Let K and L be simplicial complexes, and Δ denotes algebraic shifting. Then:*

1. *$\Delta(K)$ is a simplicial complex.*
2. *$\Delta(K) = \Delta(L)$ for L combinatorially isomorphic to K.*
3. *$f(K) = f(\Delta(K))$.*
4. *$\beta(K) = \beta(\Delta(K))$.*
5. *$\Delta(K)$ is shifted (assume $\mathrm{char(K)} = 0$ for Δ^s).*

6. $\Delta^2 = \Delta$ *(assume* $\mathrm{char(K)} = 0$ *for* Δ^s*).*

7. $\Delta(K)$ *is the same for fields with the same characteristic.*

8. *If* $L \subseteq K$ *then* $\Delta(L) \subseteq \Delta(K)$*.*

Later we will prove further properties of algebraic shifting, and will exhibit properties which hold only for one of the two versions.

Chapter 2

Algebraic Shifting and Basic Constructions on Simplicial Complexes

2.1 Shifting union of simplicial complexes

2.1.1 General unions

Problem 2.1.1 *([34], Problem 13) Given two simplicial complexes K and L, find all possible connections between $\Delta(K \cup L)$, $\Delta(K)$, $\Delta(L)$ and $\Delta(K \cap L)$.*

Exterior shifting

We look on $\bigwedge(K \cup L)$, $\bigwedge(K \cap L)$, $\bigwedge(K)$ and $\bigwedge(L)$ as subspaces of $\bigwedge(V)$ where $V = \mathrm{span}_{\mathrm{k}}\{e_1, ..., e_n\}$ and $[n] = (K \cup L)_0$. As before, the f_i's are generic linear combinations of the e_j's w.r.t. $\mathbb{F}$. Let $S \subseteq [n]$, $|S| = s$ and $1 \leq j$. First we find a connection between boundary operations on the spaces associated with K, L, $K \cap L$ and $K \cup L$ via the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigwedge^{j+s}(K \cap L) & \xrightarrow{i} & \bigwedge^{j+s} K \bigoplus \bigwedge^{j+s} L & \xrightarrow{p} & \bigwedge^{j+s}(K \cup L) & \longrightarrow & 0 \\
\downarrow & & \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \\
0 & \longrightarrow & \oplus \bigwedge^{j}(K \cap L) & \xrightarrow{\oplus i} & \oplus \bigwedge^{j} K \bigoplus \oplus \bigwedge^{j} L & \xrightarrow{\oplus p} & \oplus \bigwedge^{j}(K \cup L) & \longrightarrow & 0
\end{array} \tag{2.1}$$

where all sums $\oplus$ in the bottom sequence are taken over $\{A : A <_L S, |A| = s\}$ and $i(m) = (m, -m)$, $p((a, b)) = a + b$, $\oplus i(m) = (m, -m)$, $\oplus p((a, b)) =$

$a+b$, $f = \oplus_{A<_L S} f_A\lfloor(K\cap L)$, $g = (\oplus_{A<_L S} f_A\lfloor(K), \oplus_{A<_L S} f_A\lfloor(L))$ and $h = \oplus_{A<_L S} f_A\lfloor(K\cup L)$.

By the snake lemma, (2.1) gives rise to the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(f) & \longrightarrow & ker(g) & \longrightarrow & ker(h) & \stackrel{\delta}{\longrightarrow} & \\ coker(f) & \longrightarrow & coker(g) & \longrightarrow & coker(h) & \longrightarrow & 0 & & \end{array} \tag{2.2}$$

where δ is the connecting homomorphism. Let $(2.1')$ be the diagram obtained from (2.1) by replacing $A <_L S$ with $A \leq_L S$ everywhere, and renaming the maps by adding a superscript to each of them. Let $(2.2')$ be the sequence derived from $(2.1')$ by applying to it the snake lemma. If $\delta = 0$ in (2.2), and also the connecting homomorphism $\delta' = 0$ in $(2.2')$, then by Proposition 1.2.3 the following additive formula holds:

$$|I_S^j \cap \Delta^e(K\cup L)| = |I_S^j \cap \Delta^e(K)| + |I_S^j \cap \Delta^e(L)| - |I_S^j \cap \Delta^e(K\cap L)|. \tag{2.3}$$

Theorem 2.1.2 *Let K and L be two simplicial complexes, and let d be the dimension of $K\cap L$. For every subset A of the set of vertices $[n] = (K\cup L)_0$, the following additive formula holds:*

$$|I_A^{d+2} \cap \Delta^e(K\cup L)| = |I_A^{d+2} \cap \Delta^e(K)| + |I_A^{d+2} \cap \Delta^e(L)|. \tag{2.4}$$

Proof: Put $j = d+2$ in (2.1) and in $(2.1')$. Thus, the range and domain of f in (2.1) and of f' in $(2.1')$ are zero, hence $kerf = cokerf = 0$ and $kerf' = cokerf' = 0$, and the theorem follows. $\square$

It would be interesting to understand what extra information about $\Delta(K\cup L)$ we can derive by using more of the structure of $\Delta(K\cap L)$, and not merely its dimension. In particular, it would be interesting to find combinatorial conditions that imply the vanishing of δ in (2.2). The proof of Theorem 2.1.6 provides a step in this direction. The Mayer-Vietoris long exact sequence (e.g. [48] p.186) gives some information of this type, by the interpretation of the Betti vector using the shifted complex [8], mentioned in the Introduction.

Symmetric shifting

Let S be a monomial of degree s in the generic basis $Y = \{y_i\}_i$ w.r.t. the basis $X = \{x_i\}_i$ of $R = k[x_i : i \in (K\cup L)_0]$, and let $j > 0$ be an integer. Analogously to (2.1), the following is a commutative diagram of exact

sequences of subspaces of R:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I^{\perp}_{K\cap L})_{j+s} & \xrightarrow{i} & (I^{\perp}_{K})_{j+s} \bigoplus (I^{\perp}_{L})_{j+s} & \xrightarrow{p} & (I^{\perp}_{K\cup L})_{j+s} & \longrightarrow & 0 \\ \downarrow & & \downarrow f^* & & \downarrow g^* & & \downarrow h^* & & \downarrow \\ 0 & \longrightarrow & \oplus(I^{\perp}_{K\cap L})_{j} & \xrightarrow{\oplus i} & \oplus(I^{\perp}_{K})_{j} \bigoplus \oplus(I^{\perp}_{L})_{j} & \xrightarrow{\oplus p} & \oplus(I^{\perp}_{K\cup L})_{j} & \longrightarrow & 0 \end{array} \tag{2.5}$$

where all sums $\oplus$ in the bottom sequence are taken over $\{m : m <_L S, \deg(m) = s\}$ and $i(m) = (m, -m)$, $p((a, b)) = a + b$, $\oplus i(m) = (m, -m)$, $\oplus p((a, b)) = a + b$, and f^*, g^*, h^* are $\oplus_{m<_L S} m^*$ restricted to the relevant subspaces of R.

As in the exterior case, by the snake lemma we obtain the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(f^*) & \longrightarrow & ker(g^*) & \longrightarrow & ker(h^*) & \xrightarrow{\delta} & \\ coker(f^*) & \longrightarrow & coker(g^*) & \longrightarrow & coker(h^*) & \longrightarrow & 0 & & \end{array} \tag{2.6}$$

where δ is the connecting homomorphism, and similarly we get (2.6′) with δ' as the connecting homomorphism when applying the snake lemma to the diagram obtained from (2.5) by replacing $m <_L S$ with $m \leq_L S$ everywhere.

Under which conditions do we get $\delta = 0 = \delta'$, which provides an additive formula? Such conditions are given in the next two subsections.

We do not know whether the symmetric analogue of Theorem 2.1.2 holds or not.

2.1.2 How to shift a disjoint union?

As a corollary to Theorem 2.1.2 we get the following combinatorial formula for shifting the disjoint union of simplicial complexes:

Corollary 2.1.3 *Let* $(K\dot{\cup}L)_0 = [n], [n] \supseteq S = \{s_1 < \cdots < s_j < s_{j+1}\}$. *Then:*

$$S \in \Delta^e(K\dot{\cup}L) \Leftrightarrow s_{j+1} - s_j \leq |I^1_{\mathrm{init}_{|S|-1}(S)} \cap \Delta^e(K)| + |I^1_{\mathrm{init}_{|S|-1}(S)} \cap \Delta^e(L)|.$$

Proof: Put $d = -1$ and $A = \mathrm{init}_{|S|-1}(S)$ in Theorem 2.1.2, and by Proposition 1.2.4 we are done. □

Similarly, in the symmetric case note that $I^{\perp}_{\emptyset} = k$, hence for disjoint union and $j = 1, s > 0$ $(I^{\perp}_{K\cap L})_{j+s} = 0$ in (2.5), $ker f^* = coker f^* = 0$ in (2.6) and $ker(f^*)' = coker(f^*)' = 0$ in (2.6′). By Proposition 1.2.8, for every monomial $y \neq 1$ in the basis Y,

$$|J^1_y \cap \mathrm{GIN}(K\dot{\cup}L)| = |J^1_y \cap \mathrm{GIN}(K)| + |J^1_y \cap \mathrm{GIN}(L)|.$$

Translating this into terms of symmetric shifting, we obtain

Corollary 2.1.4 *Let* $(K\dot{\cup}L)_0 = [n], [n] \supseteq S = \{s_1 < \cdots < s_j < s_{j+1}\}$. *Then:*

$$S \in \Delta^s(K\dot{\cup}L) \Leftrightarrow s_{j+1} - s_j \leq |I^1_{\mathrm{init}_{|S|-1}(S)} \cap \Delta^s(K)| + |I^1_{\mathrm{init}_{|S|-1}(S)} \cap \Delta^s(L)|.$$

$\square$

As a corollary, we get the following nice equation, proposed by Kalai [34], for both versions of algebraic shifting (in the symmetric case assume $\mathrm{char}(k) = 0$):

Corollary 2.1.5 $\Delta(K\dot{\cup}L) = \Delta(\Delta(K)\dot{\cup}\Delta(L))$.

Proof: $S \in \Delta(K \cup L)$ iff (by Corollaries 2.1.3, 2.1.4) $s_{j+1} - s_j \leq D_K(S) + D_L(S)$ iff (by Proposition 1.2.5 and its symmetric analogue) $s_{j+1} - s_j \leq D_{\Delta(K)}(S) + D_{\Delta(L)}(S)$ iff (by Corollaries 2.1.3, 2.1.4) $S \in \Delta(\Delta(K)\dot{\cup}\Delta(L))$. $\square$

Remarks: (1) Above a high enough dimension (to be specified) all faces of the shifting of a union are determined by the shifting of its components: Let $st(K \cap L) = \{\sigma \in K \cup L : \sigma \cap (K \cap L)_0 \neq \emptyset\}$. Then $\Delta(K)$ and $\Delta(L)$ determine all faces of $\Delta(K \cup L)$ of dimension $> \dim(\mathrm{st}(K \cap L))$, by applying Corollary 2.1.5 to the subcomplex of $K \cup L$ spanned by the vertices $(K \cup L)_0 - (K \cap L)_0$, and using the basic properties Theorem 1.2.9(3,8).

(2) Let X be a $(t+l) \times (t+l)$ generic block matrix, with an upper block of size $t \times t$ and a lower block of size $l \times l$. Although we defined the shifting operator $\Delta = \Delta_A$ with respect to a generic matrix A, the definition makes sense for any nonsingular matrix (but in that case the resulting complex may not be shifted). Let $K_0 = [t]$ and $L_0 = [t+1, t+l]$. Corollary 2.1.5 can be formulated as

$$\Delta \circ \Delta_X(K\dot{\cup}L) = \Delta(K\dot{\cup}L)$$

because $\Delta_X(K\dot{\cup}L) = \Delta(K)\dot{\cup}\Delta(L)$ (on the right hand side of the equation the vertices of the two shifted complexes are considered as two disjoint sets). However, there are simplicial complexes C on $t+l$ vertices , for which $\Delta \circ \Delta_X(C) \neq \Delta(C)$. For example, let $t = l = 3$ and take the graph G of the octahedron $\{\{1\},\{4\}\} * \{\{2\},\{5\}\} * \{\{3\},\{6\}\}$. Then $\Delta \circ \Delta_X(G) \ni \{4,5\} \notin \Delta(G)$ over $k = \mathbb{R}$, for both versions of shifting.

(3) By induction, we get from Corollary 2.1.5 that:

$$\Delta(\dot{\cup}_{1 \leq i \leq l} K^i) = \Delta(\dot{\cup}_{1 \leq i \leq l} \Delta(K^i))$$

for any positive integer l and disjoint simplicial complexes K^i.

2.1.3 How to shift a union over a simplex?

In the case where $K \cap L =< \sigma >$ is a simplex and all of its subsets, we also get a formula for $\Delta(K \cup L)$ in terms of $\Delta(K)$, $\Delta(L)$ and $\Delta(K \cap L)$. This case corresponds to the topological operation called connected sum; see Example 2.1.8.

Theorem 2.1.6 *Let K and L be two simplicial complexes, where $K \cap L =< \sigma >$ is the simplicial complex consisting of the set σ and all of its subsets. For every $i > 0$ and every subset S of the set of vertices $[n] = (K \cup L)_0$, the following additive formula holds:*

$$|I_S^i \cap \Delta(K \cup_\sigma L)| = |I_S^i \cap \Delta(K)| + |I_S^i \cap \Delta(L)| - |I_S^i \cap 2^{[|\sigma|]}|.$$

This theorem gives an explicit combinatorial description of $\Delta(K \cup L)$ in terms of $\Delta(K)$, $\Delta(L)$ and $\dim(\sigma)$. In particular, any gluing of K and L along a d-simplex results in the same shifted complex $\Delta(K \cup L)$, depending only on $\Delta(K)$, $\Delta(L)$ and d.

Corollary 2.1.7 *Let K and L be simplicial complexes where $K \cap L =< \sigma >$ is a complete simplicial complex. Let $(K \cup L)_0 = [n], [n] \supseteq T = \{t_1 < \cdots < t_j < t_{j+1}\}$. Then:*

$$T \in \Delta(K \cup L) \Leftrightarrow t_{j+1} - t_j \leq D_K(T) + D_L(T) - D_{<\sigma>}(T).$$

Proof: Put $i = 1$ and $S = \text{init}_{|T|-1}(T)$ in Theorem 2.1.6, and by Proposition 1.2.4 (valid for the symmetric case as well) we are done. $\square$

Example 2.1.8 *Let $S(d, n)$ be a $(d-1)$-dimensional stacked sphere on n vertices. Then $\Delta(S(d, n)) = \text{span}\{\{1, 3, 4, .., \text{d}, \text{n}\}, \{2, 3, .., \text{d} + 1\}\}$ where* span *means taking the closure under the product partial order $<_P$ and under inclusion.*

proof: Let σ be the d-simplex and $\partial\sigma$ its boundary. Clearly $\Delta(\partial\sigma) = \partial\sigma$, hence the case $n = d + 1$ follows. Proceed by induction on n: use Corollary 2.1.7 to determine the shifting of the union $S(d, n) \cup \partial\sigma$ over a common facet. To obtain $\Delta(S(d, n + 1))$ one needs to delete from it one facet. This must be $\{2, 3, ..., d, d + 2\}$, which represent the extra top homology. $\square$

Remark: A more complicated proof of Example 2.1.8 was given very recently by Murai [49].

Proof of Theorem 2.1.6: For a simplicial complex H, let $\bar{H}$ denote the complete simplicial complex 2^{H_0}.

Exterior case: The inclusions $H \hookrightarrow \bar{H}$ for $H = K, L, <\sigma>$ induce a morphism from the commutative diagram (2.1) of K and L into the analogous commutative diagram $(\overline{2.1})$ of $\bar{K}$ and $\bar{L}$. By functoriality of the sequence of the snake lemma, we obtain the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & ker(f) & \longrightarrow & ker(g) & \longrightarrow & ker(h) & \xrightarrow{\delta} & coker(f) & \longrightarrow & ... \\ \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & ker(\bar{f}) & \longrightarrow & ker(\bar{g}) & \longrightarrow & ker(\bar{h}) & \xrightarrow{\bar{\delta}} & coker(\bar{f}) & \longrightarrow & ... \end{array} \tag{2.7}$$

where the bars indicate that $(\overline{2.1})$ is obtained from (2.1) by putting bars over all the complexes and renaming the maps by adding a bar over each map. Thus, if $\bar{\delta} = 0$ then also $\delta = 0$, which, as we have seen, implies (2.3). The fact that $\Delta(<\sigma>) = 2^{[|\sigma|]}$ completes the proof.

We show now that $\bar{\delta} = 0$. To simplify notation, assume that K and L are complete complexes whose intersection is σ (which is a complete complex). Consider (2.1) with $j = 1$. (It is enough to prove Theorem 2.1.6 for $i = 1$ as for every $i > 1$, $S \subseteq [n]$ and H a simplicial complex on $[n]$, $I_S^i \cap H = \biguplus_{T \in I_S^{i-1}} (I_T^1 \cap H)$.) Let $m = m_K + m_L \in ker(h)$ where $\text{supp}(m_K) \subseteq K \setminus <\sigma>$. By commutativity of the middle right square of (2.1), $\oplus_{A<_L S} f_A \lfloor (m_K), \oplus_{A<_L S} f_A \lfloor (m_L) \in \oplus_{A<_L S} \bigwedge^1(\sigma)$. If we show that

$$\oplus_{A<_L S} f_A \lfloor (\bigwedge^{1+|S|} \sigma) = \oplus_{A<_L S} f_A \lfloor (\bigwedge^{1+|S|} K) \bigcap \oplus_{A<_L S} \bigwedge^{1}(\sigma), \tag{2.8}$$

then there exists $m' \in \bigwedge^{1+|S|}(\sigma)$ such that $\oplus_{A<_L S} f_A \lfloor (m_K) = \oplus_{A<_L S} f_A \lfloor (m') = f(m')$, hence $\delta(m) = [f(m')] = 0$ (where $[c]$ denotes the image of c under the projection onto $coker(f)$) i.e. $\delta = 0$. (2.8) follows from the intrinsic characterization of the image of the maps it involves, given in Proposition 2.1.9. By Proposition 2.1.9, the right hand side of (2.8) consists of all $x \in \oplus_{R<_L S} \bigwedge^1 K$ that satisfy (a) and (b) of Proposition 2.1.9 which are actually in $\oplus_{A<_L S} \bigwedge^1(\sigma)$. By Proposition 2.1.9, this is exactly the left hand side of (2.8). $\square$

Symmetric case: Repeating the arguments for the exterior case, we need to show the following analogue of (2.8) for every monomial S in the basis Y of degree $s > 0$:

$$\oplus_{m<_L S} m^*((I_\sigma^\perp)_{s+1}) = \oplus_{m<_L S} m^*((I_K^\perp)_{s+1}) \bigcap \oplus_{m<_L S} (I_\sigma^\perp)_1. \tag{2.9}$$

This will follow from the intrinsic characterization of the image of the maps it involves, given in Proposition 2.1.10. $\square$

The following generalizes a result of Kalai for graphs ([31], Lemma 3.7).

Proposition 2.1.9 *Let H be a complete simplicial complex with $H_0 \subseteq [n]$, and let $S \subseteq [n]$, $|S| = s$. Then $\oplus_{R<_L S} f_R \lfloor (\bigwedge^{1+s} H)$ is the set of all $x = (x_R : R <_L S) \in \oplus_{R<_L S} \bigwedge^1 H$ satisfying the following:*

(a) For all pairs (i, R) such that $i \in R <_L S$: $< f_i, x_R >= 0$.

(b) For all pairs (A, B) such that $A <_L S, B <_L S$ and $|A \triangle B| = 2$: Denote $\{a\} = A \setminus B$ and $\{b\} = B \setminus A$. Then

$$- < f_b, x_A >= (-1)^{\mathrm{sgn}_{A\cup B}(a,b)} < f_a, x_B >$$

where $\mathrm{sgn}_{A\cup B}(a, b)$ is the number modulo 2 of elements between a and b in the ordered set $A \cup B$.

Proof: Let us verify first that every element in $\mathrm{Im} = \oplus_{R<_L S} f_R \lfloor (\bigwedge^{1+s} H)$ satisfies (a) and (b). Let $y \in \bigwedge^{1+s} H$. If $i \in R <_L S$ then $< f_i, f_R \lfloor y >=< f_i \wedge f_R, y >=< 0, y >= 0$, hence (a) holds. For $i \in T \subseteq [n]$ for some n, let $\mathrm{sgn}(i, T) = |\{t \in T : t < i\}| (\mathrm{mod}\ 2)$. If $A, B <_L S$, $\{a\} = A \setminus B$ and $\{b\} = B \setminus A$ then $- < f_b, f_A \lfloor y >= - < f_b \wedge f_A, y >= -(-1)^{\mathrm{sgn}(b,A\cup B)} < f_{A\cup B}, y >= -(-1)^{\mathrm{sgn}(b,A\cup B)}(-1)^{\mathrm{sgn}(a,A\cup B)} < f_a \wedge f_B, y >= (-1)^{\mathrm{sgn}_{A\cup B}(a,b)} < f_a, f_B \lfloor y >$, hence (b) holds.

We showed that every element of Im satisfies (a) and (b). Denote by X the space of all $x \in \oplus_{R<_L S} \bigwedge^1 H$ satisfying (a) and (b). It remains to show that $\dim(X) = \dim(\mathrm{Im})$.

Following the proof of Proposition 1.2.3, $\dim(\mathrm{Im}) = \dim(\bigwedge^{1+s} H) - \dim(\bigcap_{R<_L S} Ker_s f_R \lfloor (\bigwedge^{1+s} H) = |\{T \in \Delta(H) : |T| = s+1\}| - |\{T \in \Delta(H) : |T| = s+1, S_{(1)}^{(m)} \leq_L T\}| = |\{T \in \Delta(H) : |T| = s+1, \mathrm{init}_s(T) <_L S\}|$. Let $h = |H_0|$ and $\mathrm{sum}(T) = |\{t \in T : T \setminus \{t\} <_L S\}|$. Note that $\Delta(H) = 2^{[h]}$. Counting according to the initial s-sets, we conclude that in case $s < h$,

$$\dim(\mathrm{Im}) = |\{R : R <_L S, R \subseteq [h]\}|(h-s) - \sum\{\mathrm{sum}(T) - 1 : T \subseteq [h], |T| = s+1, \mathrm{init}_s(T) <_L S, \mathrm{sum}(T) > 1\}. \quad (2.10)$$

In case $s \geq h$, $\dim(\mathrm{Im}) = 0$.

Now we calculate $\dim(X)$. Let us observe that every $x \in X$ is uniquely determined by its coordinates x_R such that $R \subseteq [h]$: Let $i \in R \setminus [h]$, $R <_L S$. Every $j \in [h] \setminus R$ gives rise to an equation (b) for the pair $(R \cup j \setminus i, R)$ and every $j \in R \cap [h]$ gives rise to an equation (a) for the pair (j, R). Recall that x_R is a linear combination of the form $x_R = \sum_{l\in[h]} \gamma_{l,R} e_l$ with scalars $\gamma_{l,R}$. Thus, we have a system of h equations on the h variables $(\gamma_{l,R})_{l\in[h]}$ of x_R, with coefficients depending only on x_F's with $F <_L R$ (actually also $|F \cap [h]| = 1 + |R \cap [h]|$) and on the generic f_k's, $k \in [n]$. This system has a unique solution as the f_k's are generic. By repeating this argument we conclude that x_R is determined by the coordinates x_F such that $F \subseteq [h]$.

Let $x(h)$ be the restriction of $x \in X$ to its $\{x_R : R \subseteq [h], R <_L S\}$ coordinates, and let $X(h) = \{x(h) : x \in X\}$. Then $\dim(X(h)) = \dim(X)$.

Let $[a]$, $[b]$ be the matrices corresponding to the equation systems (a), (b) with variables $(\gamma_{l,T})_{l\in[h],T<_L S}$ restricted to the cases $T \subseteq [h]$ and $A, B \subseteq [h]$, respectively. $[a]$ is an $s\cdot|\{R \subseteq [h] : R <_L S\}|\times h\cdot|\{R \subseteq [h] : R <_L S\}|$ matrix and $[b]$ is a $|\{A, B \subseteq [h] : A, B <_L S, |A \triangle B| = 2\}| \times h\cdot|\{R \subseteq [h] : R <_L S\}|$ matrix.

We observe that the row spaces of $[a]$ and $[b]$ have a zero intersection. Indeed, for a fixed $R \subseteq [h]$, the row space of the restriction of $[a]$ to the h columns of R is in $\mathrm{span}_{\mathrm{k}}\{f_i^0 : i \in R\}$ (recall that f_i^0 is the obvious projection of f_i on the coordinates $\{e_j : j \in H_0\}$), and the row space of the restriction of $[b]$ to the h columns of R is in $\mathrm{span}_{\mathrm{k}}\{f_j^0 : j \in [h] \setminus R\}$. But as the f_k^0's, $k \in [h]$, are generic, $\mathrm{span}_{\mathrm{k}}\{f_k^0 : k \in [h]\} = \bigwedge^1 H$. Hence $\mathrm{span}_{\mathrm{k}}\{f_i^0 : i \in R\} \cap \mathrm{span}_{\mathrm{k}}\{f_j^0 : j \in [h] \setminus R\} = \{0\}$. We conclude that the row spaces of $[a]$ and $[b]$ have a zero intersection.

$[a]$ is a diagonal block matrix whose blocks are generic of size $s \times h$, hence

$$\mathrm{rank}([a]) = s \cdot |\{R : R <_L S, R \subseteq [h]\}| \tag{2.11}$$

in case $s < h$.

Now we compute $\mathrm{rank}([b])$. For $T \subseteq [h]$, $|T| = s + 1$, let us consider the pairs in (b) whose union is T. If (A, B) and (C, D) are such pairs, and $A \neq C$, then (A, C) is also such a pair. In addition, if A, B, C are different (the union of each two of them is T) then the three rows in $[b]$ indexed by (A, B), (A, C) and (B, C) are dependent; the difference between the first two equals the third. Thus, the row space of all pairs (A, B) with $A \cup B = T$ is spanned by the rows indexed $(\mathrm{init}_s(T), B)$ where $\mathrm{init}_s(T) \cup B = T$.

We verify now that the rows $\bigcup\{(\mathrm{init}_s(T), B) : \mathrm{init}_s(T) \cup B = T \subseteq [h], |T| = s + 1, |B| = s\}$ of $[b]$ are independent. Suppose that we have a nontrivial linear dependence among these rows. Let B' be the lexicographically maximal element in the set of all B's appearing in the rows (A, B) with nonzero coefficient in that dependence. There are at most $h - s$ rows with nonzero coefficient whose restriction to their h columns of B' is nonzero (they correspond to A's with $A = \mathrm{init}_s(B \cup \{i\}$ where $i \in [h] \setminus B$). Again, as the f_i's are generic, this means that the restriction of the linear dependence to the h columns of B' is nonzero, a contradiction. Thus,

$$\mathrm{rank}([b]) = |\bigcup_{B<_L S}\{(\mathrm{init}_s(T), B) : \mathrm{init}_s(T) \cup B = T \subseteq [h], |T| = s+1, |B| = s\}|$$

$$= \sum\{\mathrm{sum}(T) - 1 : T \subseteq [h], |T| = s+1, \mathrm{init}_s(T) <_L S, \mathrm{sum}(T) > 1\}. \tag{2.12}$$

(Note that indeed $B <_L S$ implies $\mathrm{init}_s(T) <_L S$ as $B \subseteq T$.)

For $s < h$, $\dim(X(h)) = h \cdot |\{R : R <_L S, R \subseteq [h]\}| - \mathrm{rank}([a]) - \mathrm{rank}([b])$, which by (2.10), (2.11) and (2.12) equals $\dim(\mathrm{Im})$. For $s \geq h$, $\dim(X(h)) = 0 = \dim(\mathrm{Im})$. This completes the proof. □

For symmetric shifting, the following analogous assertion holds:

Proposition 2.1.10 *Let H be a complete simplicial complex with $H_0 \subseteq [n]$, and let S be a monomial of degree s in the generic basis Y of $R_1 = k[x,..., x_n]_1$. Then $\oplus_{m<_L S} m^*((I_H^{\perp})_{s+1})$ is the set of all $x = (x_m : m <_L S) \in \oplus_{m<_L S}((I_H^{\perp})_1)$ satisfying the following:*

(*) For all pairs of monomials in Y (A, B) such that $A <_L S, B <_L S$ and $g = \gcd(A, B)$ has degree $s - 1$, denote $y_A = \frac{A}{g}$ and $y_B = \frac{B}{g}$. Then $< y_B, x_A >=< y_A, x_B >$.

Proof: For every monomial $x^I \in I_H^{\perp}$ and A, B as in (*), indeed $< y_B, A^*(x^I) >=< y_B A, x^I >=< y_A y_B g, x^I >=< y_A, B^*(x^I) >$ hence all elements in $\oplus_{m<_L S} m^*((I_H^{\perp})_{s+1})$ satisfy (*).

For the converse implication, let us compute the dimensions of both k-vector spaces. Denote for a monomial T of degree $s + 1$ in the basis Y, $\mathrm{sum}(T) := |\{i : y_i \mid T, T/y_i < S\}|$. Then:

$$\dim_k \oplus_{m<_L S} m^*((I_H^{\perp})_{s+1}) = |\{R \in GIN(H) : \mathrm{deg(R) = s{+}1, init_s(R) < S}\}| =$$

$$|H_0||\{R : R < S, \mathrm{supp(R) \subseteq H_0}\}| - \sum \{\mathrm{sumT{-}1 : \ supp(T) \subseteq H_0, init_s(T) < S, sumT > 1}\}. \tag{2.13}$$

Let $X(*)$ denote the space of elements $x = (x_m : m <_L S) \in \oplus_{m<_L S}(I_H^{\perp})_1$ satisfying (*). Note that if $i \in \mathrm{supp(R) \nsubseteq H_0}$ then the coordinate x_R is determined by the coordinates $\{x_{Ry_j/y_i} : j \in H_0\}$ via the equations (*) for the pairs $(R, Ry_j/y_i)$. Iterating this argument shows that the elements in $X(*)$ are determined by their coordinates which are supported on H_0. Some of the equations in (*) are dependent, let us find them a basis: consider the equations indexed by pairs $(\mathrm{init}_s(T), B)$ where $\mathrm{supp(T) \subseteq H_0, deg(T) = s{+}1}$, and $\mathrm{lcm(init_s(T), B) = T}$, as the rows of a matrix. We now show that they are independent: assume by contradiction the existence of a dependency, with all coefficients being nonzero, and let B' be the lexicographically maximal B in the pairs $(\mathrm{init}_s(T), B)$ which index it. Restrict the dependency to the H_0 coordinates, and to the $|H_0|$ columns indexed by B'. In the matrix that these columns form there are at most $|H_0|$ nonzero rows, corresponding to pairs with $(\mathrm{init}_s(T), B')$ where $T = y_i B'$, $i \in H_0$, and they create a generic block, hence the coefficients of these rows equal zero, a contradiction. We conclude

$$\dim_k X(*) \leq |H_0||\{R : R < S, \mathrm{supp(R) \subseteq H_0}\}| -$$

$$\sum\{\mathrm{sum}(T)-1:\ \mathrm{supp(T)}\subseteq \mathrm{H_0}, \mathrm{init_s(T)}<\mathrm{S}, \mathrm{sum(T)}>1\}. \tag{2.14}$$

As $\oplus_{m<_L S} m^*((I_H^\perp)_{s+1}) \subseteq X(*)$, combining with Equation (2.13) we conclude $\oplus_{m<_L S} m^*((I_H^\perp)_{s+1}) = X(*)$. $\square$

2.2 Shifting near cones

A simplicial complex K is called a *near cone* with respect to a vertex v if for every $j \in S \in K$ also $v \cup S \setminus j \in K$. We prove a decomposition theorem for the shifted complex of a near cone, from which the formula for shifting a cone Corollary 2.2.4 follows. As a preparatory step we introduce the Sarkaria map, modified for homology.

2.2.1 The Sarkaria map

Let K be a near cone with respect to a vertex $v = 1$. Let $e = \sum_{i\in K_0} e_i$ and let $f = \sum_{i\in K_0} \alpha_i e_i$ be a linear combination of the e_i's such that $\alpha_i \neq 0$ for every $i \in K_0$. Imitating the Sarkaria maps for cohomology [62], we get for homology the following linear maps:

$$(\bigwedge K, e_v\lfloor) \xrightarrow{U} (\bigwedge K, e\lfloor) \xrightarrow{D} (\bigwedge K, f\lfloor)$$

defined as follows: for $S \in K$

$$U(e_S) = \begin{cases} e_S - \sum_{i\in S}(-1)^{\mathrm{sgn}(i,S)} e_{v\cup S\setminus i} & \text{if } v \notin S \\ e_S & \text{if } v \in S \end{cases}$$

$$D^{-1}(e_S) = (\prod_{i\in S}\alpha_i)e_S.$$

It is justified to write D^{-1} as all the α_i's are non zero.

Proposition 2.2.1 *The maps U and D are isomorphisms of chain complexes. In addition they satisfy the following 'grading preserving' property: if $S \cup T \in K$, $S \cap T = \emptyset$, then*

$$U(e_S \wedge e_T) = U(e_S) \wedge U(e_T) \textit{ and } D(e_S \wedge e_T) = D(e_S) \wedge D(e_T).$$

Proof: The check is straight forward. First we check that U and D are chain maps. Denote $\alpha_S = \prod_{i\in S}\alpha_i$. For every e_S where $S \in K$, D satisfies

$$D \circ e\lfloor(e_S) = D(\sum_{j\in S}(-1)^{\mathrm{sgn}(j,S)} e_{S\setminus j}) = \sum_{j\in S}(-1)^{\mathrm{sgn}(j,S)}\frac{\alpha_j}{\alpha_S} e_{S\setminus j}$$

and

$$f\lfloor \circ D(e_S) = f\lfloor (\frac{1}{\alpha_S} e_S) = \sum_{j\in S} (-1)^{\mathrm{sgn}(j,S)} \frac{\alpha_j}{\alpha_S} e_{S\setminus j}.$$

For U: if $v \in S$ we have

$$U \circ e_v \lfloor (e_S) = U(e_{S\setminus v}) = e_{S\setminus v} - \sum_{i\in S\setminus v} (-1)^{\mathrm{sgn}(i,S\setminus v)} e_{S\setminus i} = \sum_{j\in S} (-1)^{\mathrm{sgn}(j,S)} e_{S\setminus j}.$$

The last equation holds because $v = 1$. Further,

$$e\lfloor \circ U(e_S) = e\lfloor (e_S) = \sum_{j\in S} (-1)^{\mathrm{sgn}(j,S)} e_{S\setminus j}.$$

If $v \notin S$ we have

$$U \circ e_v \lfloor (e_S) = U(0) = 0$$

and

$$e\lfloor \circ U(e_S) = e\lfloor (e_S) - e\lfloor (\sum_{j\in S} (-1)^{\mathrm{sgn}(j,S)} e_{S\cup v\setminus j}) =$$

$$\sum_{j\in S} (-1)^{\mathrm{sgn}(j,S)} e_{S\setminus j} - \sum_{i\in S} (-1)^{\mathrm{sgn}(i,S)} \sum_{t\in S\cup v\setminus i} (-1)^{\mathrm{sgn}(t,S\cup v\setminus i)} e_{S\cup v\setminus\{i,t\}} =$$

$$\sum_{j\in S} (-1)^{\mathrm{sgn}(j,S)} e_{S\setminus j} (1 - (-1)^{\mathrm{sgn}(v,S\cup v\setminus j)}) - \sum_{j,i\in S, i\neq j} (-1)^{\mathrm{sgn}(i,S)} (-1)^{\mathrm{sgn}(j,S\cup v\setminus i)} e_{S\cup v\setminus\{i,j\}}.$$

In the last line, the left sum is zero as $v = 1$, and for the same reason the right sum can be written as:

$$= \sum_{j,i\in S, i<j} ((-1)^{\mathrm{sgn}(i,S)+\mathrm{sgn}(j,S\setminus i)} + (-1)^{\mathrm{sgn}(j,S)+\mathrm{sgn}(i,S\setminus j)}) e_{S\cup v\setminus\{i,j\}}.$$

As $i < j$, the $\{i,j\}$ coefficient equals

$$(-1)^{\mathrm{sgn}(i,S)+\mathrm{sgn}(j,S)+1} + (-1)^{\mathrm{sgn}(j,S)+\mathrm{sgn}(i,S)} = 0,$$

hence $e\lfloor \circ U(e_S) = U \circ e_v \lfloor (e_S)$ for every $S \in K$. By linearity of U and D (and of the boundary maps), we have that U, D are chain maps. To show that U, D are onto, it is enough to show that each e_S, where $S \in K$, is in their image. This is obvious for D. For U: if $v \in S$ then $U(e_S) = e_S$, otherwise $e_S = U(e_S) + \sum_{i\in S} (-1)^{\mathrm{sgn}(i,S)} e_{v\cup S\setminus i}$, which is a linear combination of elements in $\mathrm{Im}(U)$, so $e_S \in \mathrm{Im}(U)$ as well. Comparing dimensions, U and D are also 1-1.

We now show that U 'preserves grading' in the described above sense (for D it is clear). For disjoint subsets of $[n]$ define $\mathrm{sgn}(S,T) = |\{(s,t) \in S \times T :$

$t < s\}|(\text{mod } 2)$. Let S, T be disjoint sets such that $S \cup T \in K$. By $S \cup T$ we mean the ordered union of S and T (and similarly for other set unions).

case 1: $v \notin S \cup T$.

$$U(e_S) \wedge U(e_T) =$$

$$e_S \wedge e_T + \sum_{i \in S} (-1)^{\text{sgn}(i,S)} e_{S \cup v \setminus i} \wedge e_T + \sum_{j \in T} (-1)^{\text{sgn}(j,T)} e_S \wedge e_{T \cup v \setminus j} =$$

$$(-1)^{\text{sgn}(S,T)} (e_{S \cup T} + \sum_{l \in S \cup T} (-1)^{\text{sgn}(l, S \cup T)} e_{S \cup T \cup v \setminus l}) =$$

$$U(e_S \wedge e_T),$$

where the middle equation uses the fact that $v = 1$, which leads to the following sign calculation:

$$(-1)^{\text{sgn}(i,S)} (-1)^{\text{sgn}(S \cup v \setminus i, T)} = (-1)^{\text{sgn}(i,S) + \text{sgn}(S \setminus i, T)} =$$

$$(-1)^{\text{sgn}(i,S) + \text{sgn}(S,T) + \text{sgn}(i,T)} = (-1)^{\text{sgn}(S,T)} (-1)^{\text{sgn}(i, S \cup T)}.$$

case 2: $v \in S \setminus T$.

$$U(e_S) \wedge U(e_T) = e_S \wedge (e_T - \sum_{t \in T} (-1)^{\text{sgn}(t,T)} e_{T \cup v \setminus t}) = e_S \wedge e_T = U(e_S \wedge e_T).$$

case 3: $v \in T \setminus S$. A similar calculation to the one for case 2 holds. $\square$

Remark: The 'grading preserving' property of U and D extends to the case where $S \cap T \neq \emptyset$ ($S, T \in K$), but we won't use it here. One has to check that in this case (where clearly $e_S \wedge e_T = 0$):

$$U(e_S) \wedge U(e_T) = D(e_S) \wedge D(e_T) = 0.$$

2.2.2 Shifting a near cone: exterior case.

Theorem 2.2.2 *Let K be a near cone on a vertex set $[n]$ with respect to a vertex $v = 1$. Let $X = \{f_i : 1 \leq i \leq n\}$ be some basis of $\bigwedge^1 K$ such that f_1 has no zero coefficients as a linear combination of some given basis elements e_i's of $\bigwedge^1 K$, and such that for $g_{i-1} = f_i - < f_i, e_1 > e_1$, $Y = \{g_i : 1 \leq i \leq n-1\}$ is a linearly independent set. Then:*

$$\Delta_X(K) = (1 * (\Delta_Y(\text{lk}(v, K)) + 1)) \cup B$$

where B is the set $\{S \in \Delta_X(K) : 1 \notin S\}$, $j * L := \{j \cup T : T \in L\}$, $L + j := \{T + j : T \in L\}$ and $T + j := \{t + j : t \in T\}$.

Proof: Clearly for every $l \geq 0$: $Ker_l e_v\lfloor = \bigwedge^{l+1} \mathrm{ast}(v,K)$ and $\mathrm{Im}_l\, e_v\lfloor = \bigwedge^{l} \mathrm{lk}(v,K)$. Using the Sarkaria map $D \circ U$, we get that $\mathrm{Im}\, f_1\lfloor$ is isomorphic to $\bigwedge \mathrm{lk}(v,K)$ and is contained, because of 'grading preserving', in a sub-exterior-algebra generated by the elements $b_i = DU(e_i) = \frac{1}{\alpha_i}e_i - \frac{1}{\alpha_v}e_v$, $i \in K_0 \setminus v$ (see Proposition 2.2.1). Let $S \subseteq [n], |S| = l, 1 \notin S$. Recall that $(g \wedge f)\lfloor h = g\lfloor(f\lfloor h)$. Now we are prepared to shift.

$$\bigcap_{R <_L 1 \cup S} \mathrm{Ker}_l\, f_R\lfloor \cong \mathrm{Ker}_l\, f_1\lfloor \oplus \bigcap_{1 \notin R <_L S} \mathrm{Ker}\, f_R\lfloor(\mathrm{Im}_l\, f_1\lfloor \to k),$$

which by the Sarkaria map is isomorphic to

$$\wedge^{l+1} \mathrm{ast}(v,K) \oplus \bigcap_{1 \notin T <_L S} \mathrm{Ker}(f_T\lfloor(DU(\wedge^l \mathrm{lk}(v,K)) \to k). \tag{2.15}$$

Denote by π_t the natural projection $\pi_t : \mathrm{span}_k\{e_R : |R| = t\} \to \mathrm{span}_k\{e_R : |R| = t, v \notin R\}$, and by M the matrix $(< \pi_l \circ (DU)^* f_T, e_R >)$ where $1 \notin T <_L S, R \in \mathrm{lk}(v,K)_{l-1}$. Then

$$\bigcap_{1 \notin T <_L S} \mathrm{Ker}(f_T)\lfloor(DU(\wedge^l \mathrm{lk}(v,K)) \to k) \cong \mathrm{Ker}(M)$$

Let G be the matrix $(< g_{T-1}, e_R >)$, where $1 \notin T <_L S, R \in \mathrm{lk}(v,K)_{l-1}$. Then M is obtained from G by performing the following operations: multiplying rows by nonzero scalars and adding to a row multiples of lexicographically smaller rows. Thus, restricting to the first m rows of each of these two matrices we get matrices of equal rank, for every m. In particular, $\mathrm{Ker}(M) \cong \mathrm{Ker}(G)$. Hence, using the proof of Proposition 1.2.2 (note that the proof of Proposition 1.2.2 can be applied to non-generic shifting as well), by putting $Q = T - 1$ in G we get:

$$\dim \bigcap_{R <_L 1 \cup S} Ker_l f_R\lfloor(K) = \dim \wedge^{l+1} \mathrm{ast}(v,K) + \dim \bigcap_{Q <_L S-1} Ker_{l-1} g_Q\lfloor(\mathrm{lk}(v,K)).$$

As the left summand in the right hand side is a constant independent of S, it is canceled when applying the last part of Proposition 1.2.2, and we get:

$$1 \dot\cup S \in \Delta_X(K) \;\Leftrightarrow$$

$$\dim \bigcap_{R <_L 1 \cup S} \mathrm{Ker}_l\, f_R\lfloor(K) > \dim \bigcap_{R \leq_L 1 \cup S} \mathrm{Ker}_l\, f_R\lfloor(K) \;\Leftrightarrow$$

$$\dim \bigcap_{T <_L S-1} \mathrm{Ker}_{l-1}\; g_T\lfloor(\mathrm{lk}(v,K)) > \dim \bigcap_{T \leq_L S-1} \mathrm{Ker}_{l-1}\; g_T\lfloor(\mathrm{lk}(v,K)) \;\Leftrightarrow$$

$$S-1 \in \Delta_Y(\mathrm{lk}(v,K)).$$

Thus we get the claimed decomposition of $\Delta_X(K)$. $\square$

As a corollary we get the following decomposition theorem for the generic shifted complex of a near cone.

Theorem 2.2.3 *Let K be a near cone with respect to a vertex v. Then*

$$\Delta^e(K) = (1 * \Delta^e(\mathrm{lk}(v,K))) \cup B,$$

where B is the set $\{S \in \Delta^e(K) : 1 \notin S\}$.

Proof: Apply Theorem 2.2.2 for the case where X is generic. In this case, Y is also generic, and the theorem follows .$\square$

As a corollary we get the following property [34]:

Corollary 2.2.4 $\Delta^e \circ \mathrm{Cone} = \mathrm{Cone} \circ \Delta^e$.

Proof: Consider a cone over v: $\{v\} * K$. By Theorem 2.2.3, $\{1\} * \Delta^e(K) \subseteq \Delta^e(\{v\} * K)$, but those two simplicial complexes have equal f-vectors, and hence, $\{1\} * \Delta^e(K) = \Delta^e(\{v\} * K)$.$\square$

Remarks: (1) Note that by associativity of the join operation, we get by Corollary 2.2.4: $\Delta^e(K[m] * K) = K[m] * \Delta^e(K)$ for every m, where $K[m]$ is the complete simplicial complex on m vertices.

(2) Using the notation in Theorem 2.2.2 we get: $\Delta_X \circ \mathrm{Cone} = \mathrm{Cone} \circ \Delta_Y$.

(3) Recently, it was shown in [3] that $\Delta^s \circ \mathrm{Cone} = \mathrm{Cone} \circ \Delta^s$ where the field is of characteristic zero, as was claimed by Kalai in [34].

Definition 2.2.5 *K is an $i-$near cone if there exists a sequence of simplicial complexes $K = K(0) \supset K(1) \supset \cdots \supset K(i)$ such that for every $1 \leq j \leq i$ there is a vertex $v_j \in K(j-1)$ such that $K(j) = \mathrm{ast}(v_j, K(j-1))$ and $K(j-1)$ is a near cone w.r.t. v_j.*

Remark: An equivalent formulation is that there exists a permutation $\pi : K_0 = [n] \to [n]$ such that

$$\pi(i) \in S \in K, 1 \leq l < i \Rightarrow (S \cup \pi(l) \setminus \pi(i)) \in K,$$

which is more compact but less convenient for the proof of the following generalization of Theorem 2.2.3:

Corollary 2.2.6 *Let K be an i-near cone. Then*

$$\Delta^e(K) = B \cup \biguplus_{1 \leq j \leq i} j * (\Delta^e(\mathrm{lk}(v_j, K(j-1))) + j),$$

where $B = \{S \in \Delta^e(K) : S \cap [i] = \emptyset\}$.

Proof: The case $i=1$ is Theorem 2.2.3. By induction hypothesis, $\Delta(K) = \tilde{B} \cup \biguplus_{1\leq j\leq i-1} j*(\Delta^e(K(j-1))+j)$ where $\tilde{B} = \{S \in \Delta^e(K) : S\cap[i-1] = \emptyset\}$. We have to show that

$$\{S \in \Delta^e(K) : \min\{j \in S\} = i\} = i*(\Delta^e(\mathrm{lk}(v_i, K(i-1)))+i)). \quad (2.16)$$

For $|S| = l$ with $\min\{j \in S\} = i$, we have

$$\bigcap_{R<_L S} \mathrm{Ker}_{l-1} f_R\lfloor(K) = (\bigcap_{j<i} \bigcap_{R:|R|=l, j\in R} \mathrm{Ker}_{l-1} f_R\lfloor)\cap(\bigcap_{R<_L S:\min(R)=i} \mathrm{Ker}_{l-1} f_R\lfloor). \quad (2.17)$$

By repeated application of Proposition 1.2.1, for each $j<i$,

$$\bigcap_{R:|R|=l, j\in R} \mathrm{Ker}_{l-1} f_R\lfloor = \mathrm{Ker}_{l-1} f_j.$$

Hence, (2.17) equals

$$(\bigcap_{j<i} \mathrm{Ker}_{l-1} f_j\lfloor) \cap (\bigcap_{R<_L S:\min(R)=i} \mathrm{Ker}_{l-1} f_R\lfloor) =$$

$$\bigcap_{R<_L S:\min(R)=i} \mathrm{Ker}_{l-1} f_R\lfloor(A_l),$$

where $A_l = \bigcap_{j<i} \mathrm{Ker}\, f_j\lfloor(\bigwedge^l K)$. Let $A = \oplus_l A_l$.

By repeated application of the Sarkaria map, we get that $A \cong \bigwedge K(i-1)$ as graded chain complexes. Now we will show that

$$\dim \bigcap_{R<_L S:\min(R)=i} \mathrm{Ker}_{l-1} f_R\lfloor(A) = \dim \bigcap_{R<_L S-(i-1)} \mathrm{Ker}_{l-1} f_R\lfloor(\bigwedge K(i-1)). \quad (2.18)$$

Let $\varphi : \bigwedge K(i-1) \to A$ be the Sarkaria isomorphism, and let f be generic w.r.t. the basis $\{e_i, .., e_n\}$ of $\bigwedge^1 K(i-1)$. Then $\varphi(f)$ is generic w.r.t. the basis $\{\varphi(e_i), .., \varphi(e_n)\}$ of A. We can choose a generic $\bar{f}$ w.r.t. $\{e_1, .., e_n\}$ such that $<\bar{f}, \varphi(e_j)> = <\varphi(f), \varphi(e_j)>$ for every $i \leq j \leq n$. Actually, we can do so for $n-i$ generic f_j's simultaneously (as multiplying a nonsingular matrix over a field by a generic matrix over the same field results in a generic matrix over that field). We get that

$$\bigcap_{R<_L S:min(R)=i} Ker_{l-1}\bar{f}_R\lfloor(A) = \bigcap_{R<_L S-(i-1)} Ker_{l-1}\varphi(f_R)\lfloor(A)$$

$$\cong \bigcap_{R<_L S-(i-1)} Ker_{l-1} f_R\lfloor(\bigwedge K(i-1)).$$

As both the f_i's and the $\bar{f}_i$'s are generic, $\bigcap_{R<_L S:\min(R)=i} \mathrm{Ker}_{l-1} f_R\lfloor(A) \cong \bigcap_{R<_L S:\min(R)=i} \mathrm{Ker}_{l-1} \bar{f}_R\lfloor(A)$ and (2.18) follows. By applying Theorem 2.2.3 to the near cone $K(i-1)$, we see that (2.16) is true, which completes the proof. $\Box$

From our last corollary we obtain a new proof of a well known property of algebraic shifting, proved by Kalai [33]:

Corollary 2.2.7 $\Delta^e \circ \Delta^e = \Delta^e$.

Proof: For every simplicial complex K with n vertices, $\Delta^e(K)$ is shifted, (hence an n-near cone), and so are all the $\mathrm{lk}(i, (\Delta^e K)(i-1))$'s associated to it. By induction on the number of vertices, $\Delta^e(\mathrm{lk}(i, (\Delta^e K)(i-1))) = \mathrm{lk}(i, (\Delta^e K)(i-1)) - i$ for all $1 \leq i \leq n$. Thus, applying Corollary 2.2.6 to the n-near cone $\Delta^e(K)$, we get $\Delta^e(\Delta^e(K)) = \Delta^e(K)$.$\Box$

2.3 Shifting join of simplicial complexes

Let K, L be two disjoint simplicial complexes (they include the empty set). Recall that their join is the simplicial complex $K * L = \{S \cup T : S \in K, T \in L\}$. Using the Künneth theorem with field coefficients (see [48], Theorem 58.8 and ex.3 on p.373) we can describe its homology in terms of the homologies of K and L:

$$H_i(K \times L) \cong \bigoplus_{k+l=i} H_k(K) \otimes H_l(L)$$

and the exact sequence

$$0 \to \tilde{H}_{p+1}(K * L) \to \tilde{H}_p(K \times L) \to \tilde{H}_p(K) \oplus \tilde{H}_p(L) \to 0.$$

Recalling that $\beta_i(K) = |\{S \in \Delta(K) : |S| = i+1, S \cup 1 \notin \Delta(K)\}|$ ([8] for exterior case, [28] for symmetric case), we get a description of the number of faces in $\Delta(K * L)_i$ which after union with $\{1\}$ are not in $\Delta(K * L)$, in terms of numbers of faces of that type in $\Delta(K)$ and $\Delta(L)$. In particular, if the dimensions of K and L are strictly greater than 0, the Künneth theorem implies:

$$\beta_{\dim(K*L)}(K * L) = \beta_{\dim(K)}(K)\beta_{\dim(L)}(L)$$

and hence

$$|\{S \in \Delta(K * L) : 1 \notin S, |S| = \dim(K * L) + 1\}| =$$

$$|\{S \in \Delta(K) : 1 \notin S, |S| = \dim(K)+1\}| \times |\{S \in \Delta(L) : 1 \notin S, |S| = \dim(L)+1\}|.$$

We now show that more can be said about the faces of maximal size in $K * L$ that represents homology of $K * L$:

Theorem 2.3.1 *Let $|(K*L)_0| = n$. For every $i \in [n]$*

$$|\{S \in \Delta^e(K*L) : [i] \cap S = \emptyset, |S| = \dim(K*L)+1\}| =$$

$$|\{S \in \Delta^e(K) : [i]\cap S = \emptyset, |S| = \dim(K)+1\}|\times|\{S \in \Delta^e(L) : [i]\cap S = \emptyset, |S| = \dim(L)+1\}|.$$

Proof: For a generic $f = \sum_{v\in K_0\cup L_0} \alpha_v e_v$ decompose $f = f(K) + f(L)$ with supports in K_0 and L_0 respectively. Denote $\dim(K) = k, \dim(L) = l$, so $\dim(K*L) = k+l+1$. Observe that $(f(K)\lfloor(K*L)_{k+l+1}) \cap (f(L)\lfloor(K*L)_{k+l+1}) = \{0\}$. Denote by $f\lfloor(K)$ the corresponding generic boundary operation on $\mathrm{span}_{\mathrm{k}}\{e_S : S \in K\}$, and similarly for L. Looking at $\bigwedge(K*L)$ as a tensor product $(\bigwedge K) \otimes (\bigwedge L)$ we see that $\mathrm{Ker}_{k+l+1} f(K)\lfloor$ equals $\mathrm{Ker}_k f(K)\lfloor|_{\bigwedge K} \otimes \bigwedge^{1+l} L$, and also $\mathrm{Ker}_k f(K)\lfloor|_{\bigwedge K} \cong \mathrm{Ker}_k f\lfloor(K)$, and similarly when changing the roles of K and L. Hence, we get

$$\mathrm{Ker}_{k+l+1} f\lfloor= \mathrm{Ker}_{k+l+1} f(K)\lfloor \cap \mathrm{Ker}_{k+l+1} f(L)\lfloor\cong \mathrm{Ker}_k f\lfloor(K) \otimes \mathrm{Ker}_l f\lfloor(L).$$

For the first i generic f_j's, by the same argument, we have:

$$\bigcap_{j\in[i]} \mathrm{Ker}_{k+l+1} f_j\lfloor= \bigcap_{j\in[i]} \mathrm{Ker}_{k+l+1} f_j(K)\lfloor \cap \bigcap_{j\in[i]} \mathrm{Ker}_{k+l+1} f_j(L)\lfloor\cong$$

$$\bigcap_{j\in[i]} \mathrm{Ker}_k f_j\lfloor(K) \otimes \bigcap_{j\in[i]} \mathrm{Ker}_l f_j\lfloor(L).$$

By Propositions 1.2.1 and 1.2.2 we get the claimed assertion. $\square$

For symmetric shifting, the analogous assertion to Theorem 2.3.1 is false:

Example 2.3.2 *Let each of K and L consist of three points. Thus, $K*L = K_{3,3}$ is the complete bipartite graph with 3 vertices on each side. By Theorem 2.3.1, $\{3,4\} \in \Delta^e(K_{3,3})$, but $\{3,4\} \notin \Delta^s(K_{3,3})$.*

We now deal with the conjecture ([34], Problem 12)

$$\Delta(K*L) = \Delta(\Delta(K)*\Delta(L)). \tag{2.19}$$

We give a counterexample showing that it is false even if we assume that one of the complexes K or L is shifted. Denote by ΣK the suspension of K, i.e. the join of K with the (shifted) simplicial complex consisting of two points.

Example 2.3.3 *Let B be the graph consisting of two disjoint edges. In this case $\Delta(\Sigma(B)) \setminus \Delta(\Sigma(\Delta((B))) = \{\{1,2,6\}\}$ and $\Delta(\Sigma(\Delta(B))) \setminus \Delta(\Sigma(B)) = \{\{1,3,4\}\}$, for both versions of shifting. Surprisingly, we even get that*

$$\Delta(\Sigma(B)) <_L \Delta(\Sigma(\Delta(B))), \tag{2.20}$$

where the lexicographic partial order on simplicial complexes is defined (as in [34]) by: $K \leq_L L$ iff for all $r > 0$ the lexicographically first r-face in $K \triangle L$ (if exists) belongs to K.

Conjecture 2.3.4 *For any two simplicial complexes K and L:*

$$\Delta(K * L) \leq_L \Delta(\Delta K * \Delta L).$$

Very recently Satoshi Murai announced a proof of Conjecture 2.3.4 for the exterior case [50].

Conjecture 2.3.5 *(Topological invariance.) Let K_1 and K_2 be triangulations of the same topological space. Then $\Delta(\Sigma(K_1)) <_L \Delta(\Sigma(\Delta(K_1)))$ iff $\Delta(\Sigma(K_2)) <_L \Delta(\Sigma(\Delta(K_2)))$.*

It would be interesting to find out when equation (2.19) holds. If both K and L are shifted, it trivially holds as $\Delta^2 = \Delta$. By the remark to Corollary 2.2.4 it also holds if K, say, is a complete simplicial complex.

2.4 Open problems

1. (Special case of Problem 2.1.1.) Let L be a subcomplex of K. What are the relations between $\Delta(K)$, $\Delta(L)$ and $\Delta(K \cup_L \text{Cone(L)})$?

2. Characterize the (face,Betti)-vectors of quadruples $(K, L, K \cup L, K \cap L)$ (by using shifting).

   Some necessary conditions on such vectors are given by the (f,β)-vector characterization for chains of complexes w.r.t. inclusion (Björner and Kalai [8], Duval [21]), others (linear inequalities) are given by the Mayer-Vietoris exact sequence.

3. Prove equation (2.3) is the symmetric case:

   $$|I_S^j \cap \Delta^s(K \cup L)| = |I_S^j \cap \Delta^s(K)| + |I_S^j \cap \Delta^s(L)| - |I_S^j \cap \Delta^s(K \cap L)|$$

   for every simplicial complexes K, L and all sets S and positive integers j.

4. Prove Künneth theorem with field coefficients using shifting.

5. Can one recover (part of) the cohomology ring of K by knowing the shifting of suitable complexes related to K?

6. ([34], Problem 15) Is algebraic shifting a functor? (onto a category with a useful set of maps).

Chapter 3

Algebraic Shifting and Rigidity of Graphs

3.1 Basics of rigidity theory of graphs

Asimov and Roth introduced the concept of generic rigidity of graphs [1, 2]. The presentation of rigidity here is based mainly on the one in Kalai [32].

Let $G = (V, E)$ be a simple graph. A map $f : V \to \mathbb{R}^d$ is called a *d-embedding*. It is *rigid* if any small enough perturbation of it which preserves the lengths of the edges is induced by an isometry of $\mathbb{R}^d$. Formally, f is called rigid if there exists an $\varepsilon > 0$ such that if $g : V \to \mathbb{R}^d$ satisfies $d(f(v), g(v)) < \varepsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$ (where $d(a, b)$ denotes the Euclidean distance between the points a and b).

G is called *generically d-rigid* if the set of its rigid d-embeddings is open and dense in the topological vector space of all of its d-embeddings.

Given a d-embedding $f : V \to \mathbb{R}^d$, a *stress* w.r.t. f is a function $w : E \to \mathbb{R}$ such that for every vertex $v \in V$

$$\sum_{u:\{v,u\}\in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

G is called *generically d-stress free* if the set of its d-embeddings which have a unique stress ($w = 0$) is open and dense in the space of all of its d-embeddings.

Rigidity and stress freeness can be related as follows: Let $V = [n]$, and let $\mathrm{Rig}(G, f)$ be the $dn \times |E|$ matrix associated with a d-embedding f of $V(G)$ defined as follows: for its column corresponding to $\{v < u\} \in E$ put the vector $f(v) - f(u)$ (resp. $f(u) - f(v)$) at the entries of the d rows corresponding to v (resp. u) and zero otherwise. G is generically d-stress

free if $\mathrm{Ker}(\mathrm{Rig}(G, f)) = 0$ for a generic f (i.e. for an open and dense set of embeddings). G is generically d-rigid if $\mathrm{Im}(\mathrm{Rig}(G, f)) = \mathrm{Im}(\mathrm{Rig}(K_V, f)$ for a generic f, where K_V is the complete graph on $V = V(G)$. The dimensions of the kernel and image of $\mathrm{Rig}(G, f)$ are independent of the generic f we choose; we call $R(G) = \mathrm{Rig}(G, f)$ the *rigidity matrix* of G.

$\mathrm{Im}(\mathrm{Rig}(K_V, f))$ can be described by the following linear equations: $(v_1, .., v_d) \in \bigoplus_{i=1}^{d} \mathbb{R}^n$ belongs to $\mathrm{Im}(\mathrm{Rig}(K_V, f))$ iff

$$\forall 1 \leq i \neq j \leq d \quad < f_i, v_j >=< f_j, v_i > \tag{3.1}$$

$$\forall 1 \leq i \leq d \quad < e, v_i >= 0 \tag{3.2}$$

where e is the all ones vector and f_i is the vector of the i-th coordinate of the $f(v)$'s, $v \in V$. From this description it is clear that $\mathrm{rank}(\mathrm{Rig}(K_V, f)) = dn - \binom{d+1}{2}$ (see Asimov and Roth [1] for more details).

Gluck [26] has proven that

Theorem 3.1.1 *(Gluck) The graph of a triangulated* 2*-sphere is generically* 3*-rigid. Equivalently, planar graphs are generically* 3*-stress free.*

The equivalence follows from the facts that every triangulated 2-sphere with n vertices has exactly $3n - 6$ edges (hence it is generically 3-rigid iff it is generically 3-stress free), and that every planar graph is a subgraph of a triangulated 2-sphere. Gluck's proof is based on two classical theorems: one is Cauchy's rigidity theorem (e.g. [17]), which states that any combinatorial isomorphism between two convex 3-polytopes which induces an isometry on their boundaries is actually induced by an isometry of $\mathbb{R}^3$; the other is Steinitz's theorem [70], which asserts that any polyhedral 2-sphere is combinatorially isomorphic to the boundary complex of some convex 3-polytope. Whiteley [79] has found a proof of Gluck's theorem which avoids convexity, based on vertex splitting. We summarize it below.

Lemma 3.1.2 *(Whiteley) Let G' be obtained from a graph G by contracting an edge $\{u, v\}$.*

(a)If u, v have at least $d - 1$ common neighbors and G' is generically d-rigid, then G is generically d-rigid.

(b)If u, v have at most $d - 1$ common neighbors and G' is generically d-stress free, then G is generically d-stress free.

Lemma 3.1.2 gives an alternative proof of Gluck's theorem: starting with a triangulated 2-sphere, repeatedly contract edges with exactly 2 common neighbors until the 1-skeleton of a tetrahedron is reached (it is not difficult to show that this is always possible). By Theorem 3.1.2(a) it is enough to show

that the 1-skeleton of a tetrahedron is generically 3-rigid, as is well known. (By definition, the graph of a simplex is generically d-rigid for every d, and this is also true if defining rigidity via isometries of $\mathbb{R}^d$ [1]).

We will need the following gluing lemma, due of Asimov and Roth [2].

Lemma 3.1.3 *(Asimov and Roth) (1) Let G_1 and G_2 be generically d-rigid graphs. If $G_1 \cap G_2$ contains at least d vertices, then $G_1 \cup G_2$ is generically d-rigid.*

(2) Let $G_i = (V_i, E_i)$ be generically k-stress free graphs, $i = 1, 2$ such that $G_1 \cap G_2$ is generically k-rigid. Then $G_1 \cup G_2$ is generically k-stress free.

3.2 Rigidity and symmetric shifting

Let G be the 1-skeleton of a $(d-1)$-dimensional simplicial complex K with vertex set $[n]$. We define d generic degree-one elements in the polynomial ring $A = \mathbb{R}[x_1, .., x_n]$ as follows: $\theta_i = \sum_{v\in[n]} f(v)_i x_v$ where $f(v)_i$ is the projection of $f(v)$ on the i-th coordinate, $1 \leq i \leq d$. Then the sequence $\Theta = (\theta_1, .., \theta_d)$ is an l.s.o.p. for the face ring $\mathbb{R}[K] = A/I_K$ (I_K is the ideal in A generated by the monomials whose support is not an element of K). Let $H(K) = \mathbb{R}[K]/(\Theta) = H(K)_0 \oplus H(K)_1 \oplus ...$ where (Θ) is the ideal in A generated by the elements of Θ and the grading is induced by the degree grading in A. Consider the multiplication map $\omega : H(K)_1 \longrightarrow H(K)_2$, $m \to \omega m$ where $\omega = \sum_{v\in[n]} x_v$. Lee [39] proved that

$$\dim_{\mathbb{R}} \mathrm{Ker}(\mathrm{Rig}(G, f)) = \dim_{\mathbb{R}} H(K)_2 - \dim_{\mathbb{R}} \omega(H(K)_1). \qquad (3.3)$$

Assume that G is generically d-rigid. Then $\dim_{\mathbb{R}} \mathrm{Ker}(\mathrm{Rig}(G, f)) = f_1(K) - \mathrm{rank}(\mathrm{Rig}(K_V, f)) = g_2(K) = \dim_{\mathbb{R}} H(K)_2 - \dim_{\mathbb{R}} H(K)_1$. Combining with (3.3), the map ω is injective, and hence $\dim_{\mathbb{R}}(H(K)/(\omega))_i = g_i(K)$ for $i = 2$; clearly this holds for $i = 0, 1$ as well. Hence $(g_o(K), g_1(K), g_2(K))$ is an M-sequence, i.e. the Hilbert function of a standard ring - the sequence counting the dimensions of the graded pieces of the ring by their degree. To summarize:

Theorem 3.2.1 *(Lee [39]) If a simplicial complex K has a generically $(\dim K+1)$-rigid 1-skeleton, then multiplication by a generic degree 1 element $\omega : H_1(K) \to H_2(K)$ is injective. In particular, $(g_o(K), g_1(K), g_2(K))$ is an M-sequence.*

Note that if the multiplication $\omega : H(K)_1 \to H(K)_2$ is injective, then so is the multiplication by a generic monomial of degree 1, θ_{d+1}, and vice versa. In terms of GIN, this means that $\theta_{d+1}\theta_n \in GIN(K)$, equivalently that

$\{d, n\} \in \Delta^s(K)$. To summarize, K is generically d-rigid iff $\{d, n\} \in \Delta^s(K)$. Similarly, K is generically d-stress free iff $\{d+1, d+2\} \notin \Delta^s(K)$ (iff $\omega : H(K)_1 \to H(K)_2$ is onto).

3.3 Hyperconnectivity and exterior shifting

We will describe now an exterior analogue of rigidity, namely Kalai's notion of hyperconnectivity [31]. We keep the notation from the previous section and from Chapter 1, and follow the presentation in [31].

Consider the map

$$f(d, j, K) : \bigwedge^{j+1}(K) \to \bigoplus_{1}^{d} \bigwedge^{j}(K) \quad x \mapsto (f_1 \lfloor x, ..., f_d \lfloor x).$$

The dimension of its kernel equals $|\{S \in \Delta^e K : |S| = j+1, S \cap [d] = \emptyset\}|$; it follows from Propositions 1.2.1 (with R a singleton in $[d]$) and 1.2.3. Kalai [31] called a graph G *d-hyperconnected* if $\mathrm{Im}(f(d, 1, G)) = \mathrm{Im}(f(d, 1, K_{V(G)}))$, and *$d$-acyclic* if $\mathrm{Ker}(f(d, 1, G)) = 0$. With this terminology, G is d-acyclic iff $\{d+1, d+2\} \notin \Delta^e(G)$, and is d-hyperconnected iff $\{d, n\} \in \Delta^e(G)$, where $n = |V(G)|$.

We shall prove now an exterior analogue of Lemma 3.1.2:

Lemma 3.3.1 *If G' is obtained from G by contracting an edge which belongs to at most $d-1$ triangles, and G' is d-acyclic, then so is G.*

Proof: Let $\{v, u\}$ be the edge we contract, $u \mapsto v$. Consider the $dn \times |E|$ matrix A of the map $f(d, 1, G)$ w.r.t. the standard basis, where $f_i = \sum_{j=1}^{n} \alpha_{ij} e_j$, $n = |V|$: for its column corresponding to $\{v < u\} \in E$ put the vector $(\alpha_{1u}, .., \alpha_{du})^T$ (resp. $-(\alpha_{1v}, .., \alpha_{dv})^T$) at the entries of the rows corresponding to v (resp. u) and zero otherwise.

Now replace in A each α_{iv} with α_{iu} to obtain a new matrix $\hat{A}$. It is enough to show that the columns of $\hat{A}$ are independent: As the set of $dn \times |E|$ matrices with independent columns is open (in the Euclidian topology), by perturbing the α_{iu}'s in the places where $\hat{A}$ differs from A, we may obtain new generic α_{iv}'s forming a matrix with independent columns. As for every generic choice of f_i's, the map $f(d, 1, G)$ has the same rank, we would conclude that the columns of A are independent as well.

Suppose that some linear combination of the columns of $\hat{A}$ equals zero. Let $\bar{A}$ be obtained from $\hat{A}$ by adding the rows of v to the corresponding rows of u, and deleting the rows of v. Thus, a linear combination of the columns of $\bar{A}$ with the same coefficients also equals zero. $\bar{A}$ is obtained from the matrix

of $f(d, 1, G')$ by adding a zero column (for the edge $\{v, u\}$) and doubling the columns $\{v, w\}$ which correspond to common neighbors w of v and u in G. As $\mathrm{Ker}(f(d, 1, G')) = 0$, apart from the above mentioned columns the rest have coefficient zero, and pairs of columns we doubled have opposite sign. Let us look at the submatrix of $\hat{A}$ consisting of the 'doubled' columns with vertex v and the column of $\{v, u\}$, restricted to the rows of v: it has generic coefficients, d rows and at most d columns, hence its columns are independent. Thus, all coefficients in the above linear combination are zero. $\square$

Similarly,

Lemma 3.3.2 *If G' is obtained from G by contracting an edge which belongs to at least $d-1$ triangles, and G' is d-hyperconnected, then so is G.* $\square$

We need the following exterior analogue of Lemma 3.1.3:

Lemma 3.3.3 *(Kalai [31], Theorem 4.4) Let $G_i = (V_i, E_i)$ be k-acyclic graphs, $i = 1, 2$ such that $G_1 \cap G_2$ is k-hyperconnected. Then $G_1 \cup G_2$ is k-acyclic.*

Similarly, if $G_i = (V_i, E_i)$ are k-hyperconnected graphs, $i = 1, 2$ such that $|G_1 \cap G_2| \geq k$, then $G_1 \cup G_2$ is k-hyperconnected.

We will also need the easy fact that the graph of a k-simplex is d-hyperconnected for every $k \geq d$.

3.4 Minimal cycle complexes

We shall need the concept of minimal cycle complexes, introduced by Fogelsanger [24]. We summarize his theory below.

Fix a field k (or more generally, any abelian group) and consider the formal chain complex on a ground set $[n]$, $C = (\oplus\{kT : T \subseteq [n]\}, \partial)$, where $\partial(1T) = \sum_{t \in T} \mathrm{sgn}(t, T) T \setminus \{t\}$ and $\mathrm{sgn}(t, T) = (-1)^{|\{s \in T : s < t\}|}$. Define *subchain*, *minimal d-cycle* and *minimal d-cycle complex* as follows: $c' = \sum\{b_T T : T \subseteq [n], |T| = d+1\}$ is a *subchain* of a d-chain $c = \sum\{a_T T : T \subseteq [n], |T| = d+1\}$ iff for every such T, $b_T = a_T$ or $b_T = 0$. A d-chain c is a *$d-$cycle* if $\partial(c) = 0$, and is a *minimal d-cycle* if its only subchains which are cycles are c and 0. A simplicial complex K which is spanned by the support of a minimal d-cycle is called a *minimal d-cycle complex* (over k), i.e. $K = \{S : \exists T\ S \subseteq T, a_T \neq 0\}$ for some minimal d-cycle c as above. For example, triangulations of connected manifolds without boundary are minimal cycle complexes - fix $k = \mathbb{Z}_2$ and let the cycle be the sum of all facets.

The following is the main result in Fogelsanger's thesis [24].

Theorem 3.4.1 *(Fogelsanger) For* $d \geq 3$*, every minimal* $(d-1)$*-cycle complex has a generically* d*-rigid* 1*-skeleton.*

His proof relies on the following three properties of rigidity solely: Lemmata 3.1.2 and 3.1.3 and the fact that the graph of a d-simplex is generically d-rigid. As these three properties hold for hyperconnectivity as well (see Lemmata 3.3.2 and 3.3.3 and the fact that the graph of a d-simplex is d-hyperconnected), Theorem 3.4.1 holds for hyperconnectivity as well. In terms of algebraic shifting this means

Theorem 3.4.2 *For* $n > d \geq 3$ *and every minimal* $(d-1)$*-cycle complex* K *on* n *vertices, over the field* $\mathbb{R}$*,* $\{d, n\} \in \Delta(K)$ *holds for both versions of algebraic shifting.* $\square$

3.5 Rigidity and doubly Cohen-Macaulay complexes

Definition 3.5.1 *A simplicial complex* K *is* doubly Cohen-Macaulay *(*$2-CM$ *in short) over a fixed field* k*, if it is Cohen-Macaulay and for every vertex* $v \in K$*,* $K \setminus v$ *is Cohen-Macaulay of the same dimension as* K*.*

Here $K \setminus v$ is the simplicial complex $\{T \in K : v \notin T\}$. By a theorem of Reisner [57], a simplicial complex L is Cohen-Macaulay over k iff it is pure and for every face $T \in L$ (including the empty set) and every $i < \dim(\mathrm{lk}(T, L)$, $\tilde{H}_i((\mathrm{lk}(T, L); k) = 0$.

For example, triangulated spheres are 2-CM, triangulated balls are not. A *homology sphere over* k is a simplicial complex K such that for every $F \in K$ and every i $\tilde{H}_i((\mathrm{lk}(F, K); k) \cong \tilde{H}_i((S^{\dim(\mathrm{lk}(F,K)}; k)$ where S^d is the d-dimensional sphere. Based on the fact that homology spheres are 2-CM and that the g-vector of some other classes of 2-CM complexes is known to be an M-sequence (e.g. [72]), Björner and Swartz [72] recently suspected that

Conjecture 3.5.2 *([72], a weakening of Problem 4.2.) The* g*-vector of any* 2*-CM complex is an* M*-sequence.*

We prove a first step in this direction, namely:

Theorem 3.5.3 *Let* K *be a* $(d-1)$*-dimensional* 2*-CM simplicial complex (over some field) where* $d \geq 4$*. Then* $(g_0(K), g_1(K), g_2(K))$ *is an* M*-sequence.*

This theorem follows from the following theorem, combined with Theorem 3.2.1.

Theorem 3.5.4 *Let K be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geq 3$. Then K has a generically d-rigid 1-skeleton.*

Kalai [32] showed that if a simplicial complex K of dimension ≥ 2 satisfies the following conditions then it satisfies Barnette's lower bound inequalities:

(a) K has a generically $(\dim(K)+1)$-rigid 1-skeleton.

(b) For each face F of K of codimension > 2, its link $\mathrm{lk}(F,K)$ has a generically $(\dim(\mathrm{lk}(F,K))+1)$-rigid 1-skeleton.

(c) For each face F of K of codimension 2, its link $\mathrm{lk}(F,K)$ (which is a graph) has at least as many edges as vertices.

Kalai used this observation to prove that Barnette's inequalities hold for a large class of simplicial complexes.

Observe that the link of a vertex in a 2-CM simplicial complex is 2-CM, and that a 2-CM graph is 2-connected. Combining it with Theorem 3.5.4 and the above result of Kalai we conclude:

Corollary 3.5.5 *Let K be a $(d-1)$-dimensional 2-CM simplicial complex where $d \geq 3$. For all $0 \leq i \leq d-1$ $f_i(K) \geq f_i(n,d)$ where $f_i(n,d)$ is the number of i-faces in a (equivalently every) stacked d-polytope on n vertices. (Explicitly, $f_{d-1}(n,d) = (d-1)n-(d+1)(d-2)$ and $f_i(n,d) = \binom{d}{i}n - \binom{d+1}{i+1}i$ for $1 \leq i \leq d-2$.)* $\square$

Theorem 3.5.4 is proved by decomposing K into a union of minimal $(d-1)$-cycle complexes (defined in Section 3.4). Each of these pieces has a generically d-rigid 1-skeleton by Theorem 3.4.1, and the decomposition is such that gluing the pieces together results in a complex with a generically d-rigid 1-skeleton. The decomposition is detailed in Theorem 3.5.8 below. Its proof is by induction on $\dim(K)$. Let us first consider the case where K is 1-dimensional.

A (simple finite) graph is 2-*connected* if after a deletion of any vertex from it, the remaining graph is connected and non trivial (i.e. is not a single vertex nor empty). Note that a graph is 2-CM iff it is 2-connected.

Lemma 3.5.6 *A graph G is 2-connected iff there exists a decomposition $G = \cup_{i=1}^{m} C_i$ such that each C_i is a simple cycle and for every $1 < i \leq m$, $C_i \cap (\cup_{j<i} C_j)$ contains an edge.*

Moreover, for each $i_0 \in [m]$ the C_i's can be reordered by a permutation $\sigma : [m] \to [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $C_{\sigma^{-1}(i)} \cap (\cup_{j<i} C_{\sigma^{-1}(j)})$ contains an edge.

Proof: Whitney [80] showed that a graph G is 2-connected iff it has an open ear decomposition, i.e. there exists a decomposition $G = \cup_{i=0}^{m} P_i$ such that

each P_i is a simple open path, P_0 is an edge, $P_0 \cup P_1$ is a simple cycle and for every $1 < i \leq m$ $P_i \cap (\cup_{j<i} P_j)$ equals the 2 end vertices of P_i.

Assume that G is 2-connected and consider an open ear decomposition as above. Let $C_1 = P_0 \cup P_1$. For $i > 1$ choose a simple path $\tilde{P}_i$ in $\cup_{j<i} P_j$ that connects the 2 end vertices of P_i, and let $C_i = P_i \cup \tilde{P}_i$. $(C_1, ..., C_m)$ is the desired decomposition sequence of G.

Let C be the graph whose vertices are the C_i's and two of them are neighbors iff they have an edge in common. Thus, C is connected, and hence the 'Moreover' part of the Lemma is proved.

The other implication, that such a decomposition implies 2-connectivity, will not be used in the sequel, and its proof is omitted. $\square$

For the induction step we need the following cone lemma. For v a vertex not in the support of a $(d-1)$-chain c, let $v * c$ denote the following d-chain: if $c = \sum\{a_T T : v \notin T \subseteq [n], |T| = d\}$ where $a_T \in k$ for all T, then $v * c = \sum\{\text{sgn}(v, T) a_T T \cup \{v\} : v \notin T \subseteq [n], |T| = d\}$.

Lemma 3.5.7 *Let s be a minimal $(d-1)$-cycle and let c be a minimal d-chain such that $\partial(c) = s$, i.e. c has no proper subchain c' such that $\partial(c') = s$. For v a vertex not in any face in* $\text{supp}(c)$*,the support of c, define $\tilde{s} = c - v * s$. Then $\tilde{s}$ is a minimal d-cycle.*

Proof: $\partial(\tilde{s}) = \partial(c) - \partial(v * s) = s - (s - v * \partial(s)) = 0$ hence $\tilde{s}$ is a d-cycle. To show that it is minimal, let $\hat{s}$ be a subchain of $\tilde{s}$ such that $\partial(\hat{s}) = 0$. Note that $\text{supp}(c) \cap \text{supp}(v * s) = \emptyset$.
Case 1: v is contained in a face in $\text{supp}(\hat{s})$. By the minimality of s, $\text{supp}(v * s) \subseteq \text{supp}(\hat{s})$. Thus, by the minimality of c also $\text{supp}(c) \subseteq \text{supp}(\hat{s})$ and hence $\hat{s} = \tilde{s}$.
Case 2: v is not contained in any face in $\text{supp}(\hat{s})$. Thus, $\text{supp}(\hat{s}) \subseteq \text{supp}(c)$. As $\partial(\hat{s}) = 0$ then $\partial(c - \hat{s}) = s$. The minimality of c implies $\hat{s} = 0$. $\square$

Theorem 3.5.8 *Let K be a d-dimensional 2-CM simplicial complex over a field k $(d \geq 1)$. Then there exists a decomposition $K = \cup_{i=1}^{m} S_i$ such that each S_i is a minimal d-cycle complex over k and for every $i > 1$, $S_i \cap (\cup_{j<i} S_j)$ contains a d-face.*

Moreover, for each $i_0 \in [m]$ the S_i's can be reordered by a permutation $\sigma : [m] \to [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $S_{\sigma^{-1}(i)} \cap (\cup_{j<i} S_{\sigma^{-1}(j)})$ contains a d-face.

proof: The proof is by induction on d. For $d = 1$, by Lemma 3.5.6 $K = \cup_{i=1}^{m(K)} C_i$ such that each C_i is a simple cycle and for every $i > 1$ $C_i \cap (\cup_{j<i} C_j)$

contains an edge. Define $s_i = \sum\{\mathrm{sgn}_e(i)e : e \in (C_i)_1\}$, then s_i is a minimal 1-cycle (orient the edges properly: $\mathrm{sgn}_e(i)$ equals 1 or -1 accordingly) whose support spans the simplicial complex C_i. Moreover, by Lemma 3.5.6 each C_{i_0}, $i_0 \in [m(K)]$, can be chosen to be the first in such a decomposition sequence.

For $d > 1$, note that the link of every vertex in a 2-CM simplicial complex is 2-CM. For a vertex $v \in K$, as $\mathrm{lk}(v, K)$ is 2-CM then by the induction hypothesis $\mathrm{lk}(v, K) = \cup_{i=1}^{m(v)} C_i$ such that each C_i is a minimal $(d-1)$-cycle complex and for every $i > 1$ $C_i \cap (\cup_{j<i} C_j)$ contains a $(d-1)$-face. Let s_i be a minimal $(d-1)$-cycle whose support spans C_i. As $K \setminus v$ is CM of dimension d, $\tilde{H}_{d-1}(K \setminus v; k) = 0$. Hence there exists a d-chain c such that $\partial(c) = s_i$ and $\mathrm{supp}(c) \subseteq K \setminus v$.

Take c_i to be such a chain with a support of minimal cardinality. By Lemma 3.5.7, $\tilde{s_i} = c_i - v * s_i$ is a minimal d-cycle. Let $S_i(v)$ by the simplicial complex spanned by $\mathrm{supp}(\tilde{s_i})$; it is a minimal d-cycle complex. By the induction hypothesis, for every $i > 1$ $S_i(v) \cap (\cup_{j<i} S_j(v))$ contains a d-face (containing v). Thus, $K(v) := \cup_{j=1}^{m(v)} S_j(v)$ has the desired decomposition for every $v \in K$. $K = \cup_{v \in K_0} K(v)$ as $\mathrm{st}(v, K) \subseteq K(v)$ for every v.

Let v be any vertex of K. Since the 1-skeleton of K is connected, we can order the vertices of K such that $v_1 = v$ and for every $i > 1$ v_i is a neighbor of some v_j where $1 \leq j < i$. Let $v_{l(i)}$ be such a neighbor of v_i. By the induction hypothesis we can order the $S_j(v_i)$'s such that $S_1(v_i)$ will contain $v_{l(i)}$, and hence, as K is pure, will contain a d-face which appears in $K(v_{l(i)})$ (this face contains the edge $\{v_i, v_{l(i)}\}$). The resulting decomposition sequence $(S_1(v_1), .., S_{m(v_1)}(v_1), S_1(v_2), .., S_{m(v_n)}(v_n))$ is as desired.

Moreover, every $S_j(v_{i_0})$ where $i_0 \in [n]$ and $j \in [m(v_{i_0})]$ can be chosen to be the first in such a decomposition sequence. Indeed, by the induction hypothesis $S_j(v_{i_0})$ can be the first in the decomposition sequence of $K(v_{i_0})$, and as mentioned before, the connectivity of the 1-skeleton of K guarantees that each such prefix $(S_1(v_{i_0}), .., S_{m(v_{i_0})}(v_{i_0}))$ can be completed to a decomposition sequence of K on the same $S_j(v_i)$'s. $\square$

proof of Theorem 3.5.4: Consider a decomposition sequence of K as guaranteed by Theorem 3.5.8, $K = \cup_{i=1}^{m} S_i$. By Theorem 3.4.1 each S_i has a generically d-rigid 1-skeleton. By Lemma 3.1.3 for all $2 \leq i \leq m$ $\cup_{j=1}^{i} S_j$ has a generically d-rigid 1-skeleton, in particular K has a generically d-rigid 1-skeleton $(i = m)$. $\square$

Theorem 3.5.4 follows also from the following corollary combined with Theorem 3.4.1.

Corollary 3.5.9 *Let K be a d-dimensional 2-CM simplicial complex over a field k ($d \geq 1$). Then K is a minimal cycle complex over the Abelian group $\tilde{k} = k(x_1, x_2, ...)$ whose elements are finite linear combinations of the (variables) x_i's with coefficients in k.*

Proof: Consider a decomposition $K = \cup_{i=1}^m S_i$ as guaranteed by Theorem 3.5.8, where $S_i = \overline{\text{supp}(c_i)}$, the closure w.r.t. inclusion of $\text{supp}(c_i)$, for some minimal d-cycle c_i over k. Define $\tilde{c_i} = x_i c_i$, thus $\tilde{c_i}$ is a minimal cycle over $\tilde{k}$. Define $\tilde{c} = \sum_{i=1}^m \tilde{c_i}$. Clearly $\tilde{c}$ is a cycle over $\tilde{k}$ whose support spans K. It remains to show that $\tilde{c}$ is minimal. Let $\tilde{c}'$ be a subchain of $\tilde{c}$ which is a cycle, $\tilde{c}' \neq \tilde{c}$. We need to show that $\tilde{c}' = 0$. Denote by $\tilde{\alpha_T}$ ($\tilde{\alpha_T}'$) the coefficient of the set T in $\tilde{c}$ ($\tilde{c}'$) and by $\tilde{\alpha_T}(i)$ the coefficient of the set T in $\tilde{c_i}$. If $\tilde{\alpha_T}' = 0$ then for every i such that $\tilde{\alpha_T}(i) \neq 0$, the minimality of $\tilde{c_i}$ implies that $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(i) \neq 0$. By assumption, there exists a set T_0 such that $\tilde{\alpha_{T_0}}' = 0 \neq \tilde{\alpha_{T_0}}$. In particular, there exists an index i_0 such that $\tilde{\alpha_{T_0}}(i_0) \neq 0$, hence $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(i_0) \neq 0$. As $S_{i_0} \cap (\cup_{j<i_0} S_j)$ contains a d-face in case $i_0 > 1$, repeated application of the above argument implies $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(1) \neq 0$. Repeated application of the fact that $S_i \cap (\cup_{j<i} S_j)$ contains a d-face for $i = 2, 3, ..$ and of the above argument shows that $\tilde{\alpha_F}' = 0$ whenever $\tilde{\alpha_F}(i) \neq 0$ for some $1 \leq i \leq m$, i.e. $\tilde{c}' = 0$. $\square$

A pure simplicial complex has a *nowhere zero flow* if there is an assignment of integer non-zero wights to all of its facets which forms a $\mathbb{Z}$-cycle. This generalizes the definition of a nowhere zero flow for graphs (e.g. [63] for a survey).

Corollary 3.5.10 *Let K be a d-dimensional 2-CM simplicial complex over $\mathbb{Q}$ ($d \geq 1$). Then K has a nowhere zero flow.*

Proof: Consider a decomposition $K = \cup_{i=1}^m S_i$ as guaranteed by Theorem 3.5.8. Multiplying by a common denominator, we may assume that each $S_i = \overline{supp(c_i)}$ for some minimal d-cycle c_i over $\mathbb{Z}$ (instead of just over $\mathbb{Q}$). Let N be the maximal $|\alpha|$ over all nonzero coefficients α of the c_i's, $1 \leq i \leq m$. Let $\tilde{c} = \sum_{i=1}^m (N^m)^i c_i$. $\tilde{c}$ is a nowhere zero flow for K; we omit the details. $\square$

3.6 Shifting and minors of graphs

3.6.1 Shifting can tell minors

Inspired by Lemma 3.1.2, we will show now how shifting can tell graph minors.

Theorem 3.6.1 *The following holds for symmetric and exterior shifting: for every* $2 \leq r \leq 6$ *and every graph* G*, if* $\{r-1, r\} \in \Delta(G)$ *then* G *has a* K_r *minor.*

Note that the case $r = 5$ strengthens Gluck's Theorem 3.1.1, via the interpretation of rigidity in terms of symmetric shifting (see Section 3.2).

The proof is by induction on the number of vertices, based on contracting edges satisfying the conditions of Lemma 3.1.2. We make an essential use of Mader's theorem [42] which gives an upper bound $(r-2)n - \binom{r-1}{2}$ on the number of edges in a K_r-minor free graph with n vertices, for $r \leq 7$. Indeed, Theorem 3.6.1 can be regarded as a strengthening of Mader's theorem, as $\{l+1, l+2\} \notin \Delta(G)$ implies having at most $ln - \binom{l+1}{2}$ edges, as is clear from the facts $f(G) = f(\Delta(G))$ and $\Delta(G) \subseteq \mathrm{span}(\{\mathrm{d}, \mathrm{n}\})$ (as it is shifted). This also shows that Theorem 3.6.1 fails for $r \geq 8$, as is demonstrated for $r = 8$ by $K_{2,2,2,2,2}$, and for $r > 8$ by repeatedly coning over the resulted graph for a smaller r (e.g. [65]). It would be interesting to find a proof of Theorem 3.6.1 that avoids using Mader's theorem, and derive Mader's theorem as a corollary.

A graph is *linklessly embeddable* if there exists an embedding of it in $\mathbb{R}^3$ (where vertices and edges have disjoint images) such that every two disjoint cycles of it are unlinked closed curves in $\mathbb{R}^3$. As such graph is K_6-minor free (e.g. [59], [40]), combining with Theorem 3.6.1 we conclude:

Corollary 3.6.2 *Linklessly embeddable graphs are generically* 4*-stress free.*

Let $\mu(G)$ denote the Colin de Verdière's parameter of a graph G [16]. Colin de Verdière [16] proved that a graph G is planar iff $\mu(G) \leq 3$; Lovász and Schrijver [40] proved that G is linklessly embeddable iff $\mu(G) \leq 4$. While we have seen that Theorem 3.6.1 fails for $r \geq 8$, we conjecture that Theorem 3.1.1 and Corollary 3.6.2 extend to:

Conjecture 3.6.3 *Let* G *be a graph and let* k *be a positive integer. If* $\mu(G) \leq k$ *then* G *is generically* k*-stress free.*

For $k = 1, 2, 3, 4$ this is true: Colin de Verdière [16] showed that the family $\{G : \mu(G) \leq k\}$ is closed under taking minors for every k. Note that $\mu(K_r) = r - 1$. By Theorem 3.6.1, Conjecture 3.6.3 holds for $k \leq 4$. Conjecture 3.6.3 implies

$$\mu(G) \leq k \Rightarrow e \leq kv - \tbinom{k+1}{2}$$

(where e and v are the numbers of edges and vertices in G, respectively) which is not known either.

Now we give a proof of Theorem 3.6.1 which relies on results about graph minors which are developed in the next subsection, 3.6.2.

Proof of Theorem 3.6.1: For $r = 2$ the assertion of the theorem is trivial. Suppose $K_r \not\prec G$, and contract edges belonging to at most $r-3$ triangles as long as it is possible. Denote the resulted graph by G'. Repeated application of Lemmata 3.1.2 and 3.3.1 asserts that if G' is generically $(r-2)$-stress free / $(r-2)$-hyperconnected, then so is G. In case G' has no edges, it is trivially $(r-2)$-stress free / hyperconnected. Otherwise, G' has an edge, and each edge belongs to at least $r-2$ triangles. For $2 < r < 6$, by Proposition 3.6.5 G' has a K_r minor, hence so has G, a contradiction. For $r = 6$, by Proposition 3.6.6 G' either has a K_6 minor which leads to a contradiction, or G' is a clique sum over K_r for some $r \leq 4$. In the later case, denote $G' = G_1 \cup G_2$, $G_1 \cap G_2 = K_r$. As the graph of a simplex is k-rigid/hyperconnected for any k, by Lemmata 3.1.3 and 3.3.3 it is enough to show that each G_i is generically $(r-2)$-stress free / $(r-2)$-acyclic, which follows from induction hypothesis on the number of vertices. $\square$

Remark: We can prove the case $r = 5$ avoiding Mader's theorem, by using Wagner's structure theorem for K_5-minor free graphs ([20], Theorem 8.3.4) and Lemmata 3.1.3 and 3.3.3. Using Wagner's structure theorem for $K_{3,3}$-minor free graphs ([20], ex.20 on p.185) and Lemmata 3.1.3 and 3.3.3, we conclude that $K_{3,3}$-minor free graphs are generically 4-stress free / 4-acyclic.

3.6.2 Minors

All graphs we consider are simple, i.e. with no loops and no multiple edges. Let $e = \{v, u\}$ be an edge in a graph G. By *contracting* e we mean identifying the vertices v and u and deleting the loop and one copy of each double edge created by this identification, to obtain a new (simple) graph. A graph H is called a *minor* of a graph G, denoted $H \prec G$, if by repeated contraction of edges we can obtain H from a subgraph of G. In the sequel we shall make an essential use of the following Theorem of Mader [42]:

Theorem 3.6.4 *(Mader) For $3 \leq r \leq 7$, if a graph G on n vertices has no K_r minor then it has at most $(r-2)n - \binom{r-1}{2}$ edges.*

Proposition 3.6.5 *For $3 \leq r \leq 5$: If G has an edge and each edge belongs to at least $r-2$ triangles, then G has a K_r minor.*

Proof: For $r = 3$ G actually contains K_3 as a subgraph. Let G have n vertices and e edges. Assume (by contradiction) that $K_r \not\prec G$. W.l.o.g. G is connected.

For $r = 4$, by Theorem 3.6.4 $e \leq 2n-3$ hence there is a vertex $u \in G$ with degree $d(u) \leq 3$. Denote by $N(u)$ the induced subgraph on the neighbors of

u. For every $v \in N(u)$, the edge uv belongs to at least two triangles, hence $N(u)$ is a triangle, and together with u we obtain a K_4 as a subgraph of G, a contradiction.

For $r = 5$, by Theorem 3.6.4 $e \leq 3n - 6$ hence there is a vertex $u \in G$ with degree $d(u) \leq 5$. Also $d(u) \geq 4$ (as u is not an isolated vertex). If $d(u) = 4$ then the induced subgraph on $\{u\} \cup N(u)$ is K_5, a contradiction. Otherwise, $d(u) = 5$. Every $v \in N(u)$ has degree at least 3 in $N(u)$, hence $e(N(u)) \geq \lceil 3 \cdot 5/2 \rceil = 8$. But $K_4 \not< N(u)$, hence $e(N(u)) \leq 2 \cdot 5 - 3 = 7$, a contradiction.$\square$

Proposition 3.6.6 *If G has an edge and each edge belongs to at least 4 triangles, then either G has a K_6 minor, or G is a clique sum over K_r for some $r \leq 4$ (i.e. $G = G_1 \cup G_2, G_1 \cap G_2 = K_r$, $G_i \neq K_r$, $i = 1, 2$).*

Proof: We proceed as in the proof of Proposition 3.6.5: Assume that $K_6 \not< G$. W.l.o.g. G is connected. By Theorem 3.6.4 $e \leq 4n - 10$ hence there is a vertex $u \in G$ with degree $d(u) \leq 7$, also $d(u) \geq 5$. If $d(u) = 5$ then $N(u) = K_5$, a contradiction. Actually, $N(u)$ is planar: since $N(u)$ has at most 7 vertices, each of degree at least 4, if $N(u)$ were not 4-connected, it must have exactly 7 vertices and two disjoint edges such that each of their 4 vertices is adjacent to the remaining 3 vertices of $N(u)$ (whose removal disconnect $N(u)$); but such graph has a K_5 minor. As $K_5 \not< N(u)$, $N(u)$ is 4-connected. Now Wagner's structure theorem for K_5-minor free graphs ([20], Theorem 8.3.4) asserts that $N(u)$ is planar.

If $d(u) = 6$, then $12 = 3 \cdot 6 - 6 \geq e(N(u)) \geq 4 \cdot 6/2 = 12$ hence $N(u)$ is a triangulation of the 2-sphere S^2. If $d(u) = 7$, then $15 = 3 \cdot 7 - 6 \geq e(N(u)) \geq 4 \cdot 7/2 = 14$. We will show now that $N(u)$ cannot have 14 edges, hence it is a triangulation of S^2: Assume that $N(u)$ has 14 edges, so each of its vertices has degree 4, and $N(u)$ is a triangulation of S^2 minus an edge. Let us look to the unique square (in a planar embedding) and denote its vertices by A. The number of edges between A and $N(u) \setminus A$ is 8. Together with the 4 edges in the subgraph induced by A, leaves two edges for the subgraph induced by $N(u) \setminus A = \{a, b, c\}$; let a be their common vertex. We now look at the neighborhood of a in a planar embedding (it is a 4-cycle): b, c must be opposite in this square as $\{b, c\}$ is missing. Hence for $v \in A \cap N(a)$ we get that v has degree 5, a contradiction.

Now we are left to deal with the case where $N(u)$ is a triangulation of S^2, and hence a maximal K_5-minor free graph. If G is the cone over $N(u)$ with apex u, then every edge in $N(u)$ belongs to at least 3 triangles in $N(u)$. By Proposition 3.6.5, $N(u)$ has a K_5 minor, a contradiction. Hence there exists a vertex $w \neq u$, $w \in G \setminus N(u)$. Denote by $[w]$ the set of all vertices

in G connected to w by a path disjoint from $N(u)$. Denote by $N'(w)$ the induced graph on the vertices in $N(u)$ that are neighbors of some vertex in $[w]$. If $N'(w)$ is not a clique, there are two non-neighbors $x, y \in N'(w)$, and a path through vertices of $[w]$ connecting them. This path together with the cone over $N(u)$ with apex u form a subgraph of G with a K_6 minor, a contradiction.

Suppose $N'(w)$ is a clique (it has at most 4 vertices, as $N(u)$ is planar). Then G is a clique sum of two graphs that strictly contain $N'(w)$: Let G_1 be the induced graph on $[w] \cup N'(w)$ and let G_2 be the induced graph on $G \setminus [w]$. Then $G = G_1 \cup G_2$ and $G_1 \cap G_2 = N'(w)$. $\square$

Remark: In view of Theorem 3.6.4 for the case $r = 7$, we may expect the following to be true:

Problem 3.6.7 *If G has an edge and each edge belongs to at least 5 triangles, then either G has a K_7 minor, or G is a clique sum over K_l for some $l \leq 6$.*

If true, it extends the assertion of Theorem 3.6.1 to the case $r = 7$. We could show only the weaker assertion

$$G \text{ has a generic } 5-\mathrm{stress} \Rightarrow \mathrm{K}_7^- \prec \mathrm{G},$$

where K_7^- is K_7 minus an edge, by using similar arguments to those used in this section.

3.6.3 Shifting and embedding into 2-manifolds

Theorem 3.1.1 may be extended to other 2-manifolds as follows:

Theorem 3.6.8 *Let $M \neq S^2$ be a compact connected 2-manifold without boundary, and let G be a graph. Suppose that $\{r-1, r\} \in \Delta(G)$ and K_r can not be embedded in M. Then G can not be embedded in M.*

Proof: Let g be the genus of M, then $g > 0$ (e.g. the torus has genus 1, the projective plane has genus 1/2). Assume by contradiction that G embeds in M. By looking at the rigidity matrix we note that deleting from G a vertex of degree at most $r-2$ preserves the existence of $\{r-1, r\}$ in the shifted graph. Deletion preserves embeddability in M as well. Thus we may assume that G has minimal degree $\delta(G) \geq r-1$. By Euler formula $e \leq 3v - 6 + 6g$ (where e and v are the numbers of edges and vertices in G respectively). Also $e \geq (r-1)v/2$, hence $v \leq \frac{12g-12}{(r-1)-6}$. Thus $(r-1)^2 - 5(r-1) + (6-12g) \leq 0$ which implies $r \leq (7 + \sqrt{1+48g})/2$. As K_r can not be embedded in M, by Ringel and Youngs [58] proof of Heawood's map-coloring conjecture $r > (7 + \sqrt{1+48g})/2$, a contradiction.$\square$

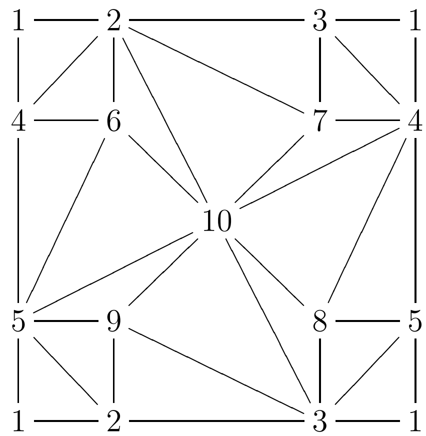

Figure 3.1: Linkless graph of a torus

Remark: For any compact connected 2-manifold without boundary of positive genus, M, if M is embedded in $\mathbb{R}^3$ two linked simple closed curves on it exist. One may ask whether the graph of any triangulated such M is always not linkless.

For the projective plane this is true. It follows from the fact that the two minimal triangulations of the projective plane (w.r.t. edge contraction), determined by Barnette [5], have a minor from the Petersen family, and hence are not linkless, by the result of Robertson, Seymour and Thomas [59]. Moreover, the graph of any polyhedral map of the projective plane is not linkless, as its 7 minimal polyhedral maps (w.r.t. edge contraction), determined by Barnette [4], have graphs equal to 6 of the members in Petersen family.

Examining the 21 minimal triangulations of the torus, see Lavrenchenko [38], we note that 20 of them have a K_6 minor, and hence are not linkless, but the last one is linkless, see Figure 4.1 (one checks that it contains no minor from Petersen's family). Taking connected sums of this triangulation, we obtain linkless graphs triangulating any oriented surface of positive genus. By performing stellar operations we obtain linkless graphs with arbitrarily many vertices triangulating any oriented surface of positive genus.

3.7 Open problems

1. Can the S_i's in Theorem 3.5.8 be taken to be homology spheres?

2. Can the intersections in Theorem 3.5.8 be guaranteed to be CM?

   In view of Proposition 4.3.1, if the intersections $S_i \cap (\cup_{j<i} S_j)$ in Theorem 3.5.8 can be taken to be CM, and the S_i's can be taken to be homology

spheres, then Conjecture 3.5.2 would be reduced to the conjecture that homology spheres have the weak-Lefschetz property; see Conjecture 4.1.2(2).

3. Must a graph with a generic $(r-2)$-stress contain a subdivision of K_r for $2 \leq r \leq 6$?

   The answer is positive for $r = 2, 3, 4$ as in this case G has a K_r minor iff G contains a subdivision of K_r ([20], Proposition 1.7.2). Mader proved that every graph on n vertices with more than $3n-6$ edges contains a subdivision of K_5 [43]. A positive answer in the case $r = 5$ would strengthen this result.

4. Let G be a graph and let k be a positive integer. Show that $\mu(G) \leq k$ implies that G is generically k-stress free.

5. Assume that G has an edge and each edge belongs to at least 5 triangles. Show that either G has a K_7 minor, or G is a clique sum over K_l for some $l \leq 6$.

   If true, it implies that $\{6, 7\} \in \Delta(G)$ forces a K_7 minor in G.

6. Is the graph of a triangulated non orientable 2-manifold always not linkless?

7. Prove Charney-Davis conjecture [14] for clique 3-spheres using rigidity (shifting) arguments in order to give a simpler proof than in [18]. We repeat their conjecture: Let K be a $(d-1)$ dimensional clique (homology) sphere (that is, all its missing faces are 1 dimensional), where d is even. Show that $(-1)^{\frac{d}{2}} \sum_{i=0}^{d} h_i(K) \geq 0$. Equivalently, $\sum_{0 \leq k \leq \frac{d}{2}} (-1)^{\frac{d}{2}-k} g_k(K) \geq 0$. In case $d = 4$, the conjecture reads $f_1(K) \geq 5f_0(K) - 16$ (to be compared with the LBT for spheres: $f_1(K) \geq 4f_0(K) - 10$).

Chapter 4

Lefschetz Properties and Basic Constructions on Simplicial Spheres

4.1 Basics of Lefschetz properties

Our motivating problem is the following well known McMullen's g-conjecture for spheres. Recall that by *homology sphere* (or *Gorenstein* complex*) we mean a pure simplicial complex L such that for every face $F \in L$ (including the empty set), $\mathrm{lk(F, L)}$ has the same homology (say with integer coefficients) as of a $\mathrm{dim(lk(F, L))}$-sphere.

Conjecture 4.1.1 *(McMullen [45]) Let L be a homology sphere, then its g-vector is an M-sequence.*

An algebraic approach to this problem is to associate with L a standard ring whose Hilbert function is $g(L)$, the g-vector of L. This was worked out successfully by Stanley [68] in his celebrated proof of Conjecture 4.1.1 for the case where L is the boundary complex of a simplicial polytope. The hard-Lefschetz theorem for toric varieties associated with rational polytopes, translates in this case to the following property of face rings, called *hard-Lefschetz*.

Let K be a simplicial complex on the vertex set $[n]$. Let $A = \mathbb{R}[x_1, .., x_n]$ be the polynomial ring, each variable has degree one. Recall that the face ring of K is $\mathbb{R}[K] = A/I_K$ where I_K is the ideal in A generated by the monomials whose support is not an element of K. Let $\Theta = (\theta_1, .., \theta_d)$ be an l.s.o.p. of $\mathbb{R}[K]$ - it exists, e.g. [69], Lemma 5.2, and generic 1-forms $y_1, ..., y_d$ from the basis Y of A_1 (recall from subsection 1.2.2) will do. Denote

$H(K) = H(K, \Theta) = \mathbb{R}[K]/(\Theta) = H(K)_0 \oplus H(K)_1 \oplus ...$ where the grading is induced by the degree grading in A, and (Θ) is the ideal in $\mathbb{R}[K]$ generated by the images of the elements of Θ under the projection $A \to \mathbb{R}[K]$. K is called *Cohen-Macaulay* (CM for short) over $\mathbb{R}$ if for an (equivalently, every) l.s.o.p. Θ, $\mathbb{R}[K]$ is a free $\mathbb{R}[\Theta]$-module. If K is CM then $\dim_{\mathbb{R}} H(K)_i = h_i(K)$. (The converse is also true: h is an M-vector iff $h = h(K)$ for some CM complex K [69], Theorem 3.3.) For K a CM simplicial complex with symmetric h-vector, if there exists an l.s.o.p. Θ and an element $\omega \in A_1$ such that the multiplication maps $\omega^{d-2i} : H(K, \Theta)_i \longrightarrow H(K, \Theta)_{d-i}$, $m \mapsto \omega^{d-2i}m$, are isomorphisms for every $0 \leq i \leq \lfloor d/2 \rfloor$, we say that K has the *hard-Lefschetz property*, or that K is HL.

As was shown by Stanley [68], for K the boundary complex of a simplicial d-polytope P, the l.s.o.p Θ induced by the embedding of P_0 in $\mathbb{R}^d$ and $\omega = \sum_{1 \leq i \leq n} x_i$ demonstrate that K is HL; hence so do generic $y_1, ..., y_{d+1} \in Y$. In terms of GIN this is equivalent to requiring that non of the monomials $y_{d+1}^{d-2k-1} y_{d+2}^{k+1}$ are in $GIN(K)$, where $k = 0, 1, ...$ Indeed, these monomials are not in $GIN(K)$ iff the maps $y_{d+1}^{d-2i} : H(K)_i \longrightarrow H(K)_{d-i}$ are onto, and when $h(K)$ is symmetric this happens iff these maps are isomorphisms.

Let us translate the hard-Lefschetz property from terms of GIN into terms of symmetric shifting, as in [34]. Let $\Delta(d, n)$ be the pure $(d-1)$-dimensional simplicial complex with set of vertices $[n]$ and facets $\{S : S \subseteq [n], |S| = d,\ k \notin S \Rightarrow [k+1, d-k+2] \subseteq S\}$. Equivalently, $\Delta(d, n)$ is the maximal pure $(d-1)$-dimensional simplicial complex with vertex set $[n]$ which does not contain any of the sets $T_d, ..., T_{\lceil d/2 \rceil}$, where

$$T_{d-k} = \{k+2, k+3, ..., d-k, d-k+2, d-k+3, ..., d+2\},\ 0 \leq k \leq \lfloor d/2 \rfloor. \quad (4.1)$$

Note that $\Delta(d, n) \subseteq \Delta(d, n+1)$, and define $\Delta(d) = \cup_n \Delta(d, n)$. Kalai refers to the relation

$$\Delta(K) \subseteq \Delta(d) \quad (4.2)$$

as the *shifting theoretic upper bound theorem*. Using the map from $GIN(K)$ to $\Delta^s(K)$, we have just seen that for CM $(d-1)$-dimensional complexes with symmetric h-vector, $\Delta^s(K) \subseteq \Delta(d)$ is equivalent to K being HL.

To justify the terminology in (4.2), note that the boundary complex of the cyclic d-polytope on n vertices, denoted by $C(d, n)$, satisfies $\Delta^s(C(d, n)) = \Delta(d, n)$. This follows from the fact that $C(d, n)$ is HL. Recently Murai [49] proved that also $\Delta^e(C(d, n)) = \Delta(d, n)$, as was conjectured by Kalai [34]. It follows that if K has n vertices and (4.2) holds, then the f-vectors satisfy $f(K) \leq f(C(d, n))$ componentwise.

For K as above, weaker than the hard-Lefschetz property is to require only that multiplications $y_{d+1} : H(K)_{i-1} \longrightarrow H(K)_i$ are injective for $1 \leq$

$i \le \lceil d/2 \rceil$ and surjective for $\lceil d/2 \rceil < i \le d$, called here *unimodal weak-Lefschetz* property (sometimes it is called weak-Lefschetz in the literature). Even weaker is just to require that multiplications $y_{d+1} : H(K)_{i-1} \longrightarrow H(K)_i$ are injective for $1 \le i \le \lfloor d/2 \rfloor$, which we refer to as the *weak-Lefschetz property*, and say that K is WL. (Injectivity for $i \le \lceil d/2 \rceil$ in the case of Gorenstein* complexes implies also surjective maps for $\lceil d/2 \rceil < i \le d$; see the proof of Theorem 4.5.2 below.) This is equivalent to the following, in the case of symmetric shifting [9]:

$$\begin{aligned} &(1)\ S \in \Delta(K), |S| = k \ \Rightarrow\ [d-k] \cup S \in \Delta(K), \\ &(2)\ S \in \Delta(K), |S| = k < \lfloor d/2 \rfloor \ \Rightarrow\ \{d-k+1\} \cup S \in \Delta(K). \end{aligned} \tag{4.3}$$

Condition (1) holds when K is CM, and condition (2) holds iff K is WL. As was noticed in [9], (4.3) is implied by requiring that $\Delta(K)$ is pure and every $S \in \Delta(K)$ of size less than $\lfloor d/2 \rfloor$ is contained in at least 2 facets of $\Delta(K)$.

Note that if L is a homology sphere, it is in particular CM with a symmetric h-vector. If in addition it has the weak-Lefschetz property, then in the standard ring $S(L) = \mathbb{R}[K]/(\Theta, y_{d+1}, A_{1+\lfloor d/2 \rfloor}) = H(L,\Theta)/(y_{d+1}, H_{1+\lfloor d/2 \rfloor}) = S_0 \oplus S_1 \oplus ...$ the following holds: $g_i(L) = \dim_{\mathbb{R}} S_i$ for all $0 \le i \le \lfloor d/2 \rfloor$, and Conjecture 4.1.1 holds for L.

We summarize the discussion above in the following hierarchy of conjectures, where assertion (i) implies assertion $(i+1)$:

Conjecture 4.1.2 *Let L be a homology $(d-1)$-sphere. Then:*

(1) If $S \in \Delta(L)$, $|S| = k \le \lfloor d/2 \rfloor$ and $S \cap [d-k+1] = \emptyset$ then $S \cup [k+2, d-k+1] \in \Delta(L)$.

This is equivalent to $\Delta(K) \subseteq \Delta(d)$, and in the symmetric case this is equivalent to L being HL.

(2) If $S \in \Delta(L)$, $|S| = k < \lfloor d/2 \rfloor$ and $S \cap [d-k+1] = \emptyset$ then $S \cup [\lceil d/2 \rceil + 2, d-k+1] \in \Delta(L)$. In the symmetric case this is equivalent to L being WL.

(3) $g(L)$ is an M-vector.

4.2 Hard Lefschetz and join

Let S be a Cohen-Macaulay $(d-1)$-simplicial complex over a field k. If there exists a degree one element ω such that multiplication

$$\omega^{d-2i} : H(S)_i \to H(S)_{d-i} \tag{4.4}$$

is an isomorphism (for some l.s.o.p.) we say that S is *i-Lefschetz* and that ω is an *i-Lefschetz element* of $H(S)$. If (4.4) holds for every $0 \le i \le d/2$ then S is *HL* and ω is an *HL-element* of $H(S)$.

Let us recall a few ring theoretic terms, see e.g. [69] for details. Let $A = k[x_1, ..., x_n]$. An A-module M of dimension d is *Cohen-Macaulay* (CM) if for generic (i.e., for some algebraically independent) $y_1, ..., y_d \in A_1$ M is a free module over the subring $k[y_1, .., y_d]$. If in addition $\dim_k \operatorname{soc} M/(y_1, ..., y_d)M = 1$ where $\operatorname{soc} M := \{u \in M : A_+ u = 0\}$, $A_+ = A_1 \oplus A_2 \oplus ...$ and $(y_1, ..., y_d)$ is the obvious ideal in A, then M is *Gorenstein*. Note that Gorenstein* complexes have Gorenstein face rings (as A-modules).

Lemma 4.2.1 *Let M be a d-dimensional Gorenstein module over the polynomial ring $R = \mathbb{R}[x_1, ..., x_n]$, with an l.s.o.p. Θ. Denote $H = M/(\Theta)M$. Then for every $0 \leq i \leq d/2$ the pairing $H_i \times H_{d-i} \to \mathbb{R}$, $(x, y) \mapsto \alpha(xy)$ is non-degenerated under any fixed isomorphism $\alpha : H_d \cong \mathbb{R}$.*

Proof: M is Gorenstein, so, by definition, $\dim_{\mathbb{R}} \mathrm{socH} = 1$. As M is d-dimensional, $\dim_{\mathbb{R}} \mathrm{H_d} \geq 1$, but $H_d \subseteq \mathrm{socH}$, thus $\mathrm{socH} = \mathrm{H_d}$. As R_+ is generated by $\{x_1, ..., x_n\}$, we get that for every $0 \leq i < d$ and $0 \neq u \in H_i$ there exists x_j such that $x_j u \neq 0$, and inductively there exists a monomial m of degree $d - i$ such that $mu \neq 0$, thus the pairing is non-degenerated. $\square$

Lemma 4.2.2 *Let K be a $(d-1)$-dimensional Gorenstein* complex with an l.s.o.p. Θ and an HL element ω over the reals. Let $H = \mathbb{R}[K]/(\Theta)$ and fix an isomorphism $\alpha : H_d \cong \mathbb{R}$. Then for every $0 \leq i \leq d/2$ there is an induced non degenerated bilinear form on H_i given by $< x, y >= \alpha(\omega^{d-2i} xy)$.*

Proof: Clearly $<,>$ is bilinear and symmetric. For $0 \neq x \in H_i$, by assumption $0 \neq \omega^{d-2i} x \in H_{d-i}$, and by Lemma 4.2.1 there exists $y \in H_i$ such that $\alpha(\omega^{d-2i} xy) \neq 0$, hence $<,>$ is not degenerated. $\square$

Lemma 4.2.3 *Under the assumptions of Lemma 4.2.2, H decomposes into a direct sum of $\mathbb{R}[\omega]$-invariant spaces, each is of the form*

$$V_m = \mathbb{R}m \oplus \mathbb{R}\omega m \oplus ... \oplus \mathbb{R}\omega^{d-2i} m$$

for $m \in \mathbb{R}[K]/(\Theta)$ of degree i for some $0 \leq i \leq d/2$.

Proof: V_1 $(1 \in H_0)$ is an $\mathbb{R}[\omega]$-invariant space which contain H_0. Assume that for $1 \leq i \leq d/2$ we have already constructed a direct sum of $\mathbb{R}[\omega]$-invariant spaces, $\tilde{V}_{i-1}$, which contains $\tilde{H}_{i-1} := H_0 \oplus ... \oplus H_{i-1}$, in which each V_m contains some nonzero element of $\tilde{H}_{i-1}$. We now extend the construction to have these properties w.r.t. $\tilde{H}_i$. By assumption $\omega H_{i-1} \subseteq \tilde{V}_{i-1} \cap H_i$. Let $m_1, ..., m_t \in H_i$ form a basis to the subspace of H_i orthogonal to ωH_{i-1} w.r.t. the inner product from Lemma 4.2.2. Let

$$\tilde{V}_i = \tilde{V}_{i-1} + V_{m_1} + ... + V_{m_t}. \tag{4.5}$$

We first show that each V_{m_j} is $\mathbb{R}[\omega]$-invariant, i.e. that $\omega^{d-2i+1}m_j = 0$ for $1 \leq j \leq t$. By Lemma 4.2.1 it is enough to show that for every $x \in H_{i-1}$ $\alpha(x\omega^{d-2i+1}m_j) = 0$. As $\alpha(x\omega^{d-2i+1}m_j) =< \omega x, m_j >$ indeed it equals zero.

Next we show that the sum in (4.5) is direct. As $m_1, ..., m_t \in H_i$ are linearly independent and $\omega^{d-2i} : H_i \rightarrow H_{d-i}$ is injective, then the sum $W_i := V_{m_1} + ... + V_{m_t}$ is direct. To show that $\tilde{V}_{i-1} \oplus W_i$, we check that for every $i \leq l \leq d-i$ $H_l \cap \tilde{V}_{i-1} \cap W_i = 0$ (for $l < i$ and for $l > d-i$ $W_i \cap H_l = 0$). Indeed, an element in the intersection is of the form $\omega^{l-i}\omega x = \omega^{l-i}y$ where $x \in H_{i-1}$ and $y \in H_i$ is orthogonal to ωx. Injectivity of $\omega^{l-i} : H_i \rightarrow H_l$ implies $y = \omega x$, which equals zero by orthogonality.

As the h-vector of K is symmetric, $\tilde{V}_{\lfloor d/2 \rfloor} = H$, giving the desired decomposition. $\square$

Remark: Even if K is not HL we still get a decomposition into a direct sum of irreducible $\mathbb{R}[\omega]$-invariant spaces $V_m = \mathbb{R}m \oplus \mathbb{R}\omega m \oplus ... \oplus \mathbb{R}\omega^l m$, but no longer $l = d - 2\deg(m)$.

Theorem 4.2.4 *(With Eric Babson) Let K and L be Gorenstein* complexes on disjoint sets of vertices, of dimensions $d_K - 1, d_L - 1$, with l.s.o.p's Θ_K, Θ_L and HL elements ω_K, ω_L respectively; over the reals. Then:*

(0) $K * L$ has a symmetric h-vector and dimension $d_K + d_L - 1$.

(1) $\Theta_K \biguplus \Theta_L$ is an l.s.o.p for $K * L$ (over $\mathbb{R}$).

(2) $\omega_K + \omega_L$ is an HL element of $\mathbb{R}[K * L]/(\Theta_K \biguplus \Theta_L)$.

Proof : The h-polynomials satisfy $h(t, K * L) = h(t, K)h(t, L)$, hence the symmetry of $h(t, K * L)$ follows from that of $h(t, K)$ and $h(t, L)$:

$$t^{d_K+d_L}h(\frac{1}{t}, K * L) = t^{d_K}h(\frac{1}{t}, K)t^{d_L}h(\frac{1}{t}, L) = h(t, K)h(t, L) = h(t, K * L).$$

For a set I let $A_I := \mathbb{R}[x_i : i \in I]$ be a polynomial ring. The isomorphism $A_{K_0} \bigotimes_{\mathbb{R}} A_{L_0} \cong A_{K_0 \biguplus L_0}$, $a_K \otimes a_L \mapsto a_K a_L$ induces a structure of an $A = A_{K_0 \biguplus L_0}$ module on $\mathbb{R}[K] \bigotimes_{\mathbb{R}} \mathbb{R}[L]$, isomorphic to $\mathbb{R}[K * L]$, by $m_K \otimes m_L \mapsto m_K m_L$ and $(a_K \otimes a_L)(m_K \otimes m_L) = a_K m_K \otimes a_L m_L$. (E.g. $a_K \in A_{K_0} \subseteq A$ acts like $a_K \otimes 1$ on $\mathbb{R}[K] \bigotimes_{\mathbb{R}} \mathbb{R}[L]$.)

The above isomorphism induces an isomorphism of A-modules

$$\mathbb{R}[K * L]/(\Theta_K \biguplus \Theta_L) \cong \mathbb{R}[K]/(\Theta_K) \bigotimes_{\mathbb{R}} \mathbb{R}[L]/(\Theta_L), \tag{4.6}$$

proving (1). Actually, $\mathbb{R}[K*L]$ is both a finitely generated and free $\mathbb{R}[\Theta_K \biguplus \Theta_L]$-module, by Cohen-Macaulayness.

By Lemma 4.2.3, $\mathbb{R}[K]/(\Theta_K)$ decomposes into a direct sum of $\mathbb{R}[\omega_K]$-invariant spaces, each is of the form $V_m = \mathbb{R}m \bigoplus \mathbb{R}\omega_K m \bigoplus ... \bigoplus \mathbb{R}\omega_K^{d_K-2i}m$

for $m \in \mathbb{R}[K]/(\Theta_K)$ of degree i for some $0 \leq i \leq d_K/2$; and similarly for $\mathbb{R}[L]/(\Theta_L)$.

The $\mathbb{R}[\omega_K]$-module V_m is isomorphic to the $\mathbb{R}[\omega]$-module $\mathbb{R}[\partial\sigma^{d_K-2i}]/(\theta)$ by $\omega_K \mapsto \omega$ and $m \mapsto 1$, where σ^j is the j-simplex, θ is an l.s.o.p. induced by the positions of the vertices in an embedding of σ^{d_K-2i} as a full dimensional geometric simplex in $\mathbb{R}^{d_K-2i}$ with the origin in its interior, and $\omega = \sum_{v\in\sigma_0} x_v$ is an HL element for $\mathbb{R}[\partial\sigma^{d_K-2i}]/(\theta)$. Thus, to prove (2) it is enough to prove it for the join of boundaries of two simplices with l.s.o.p.'s as above and the HL elements having weight 1 on each vertex of the ground set.

Note that the join $\partial\sigma^k * \partial\sigma^l$ is combinatorially isomorphic to the boundary of the polytope $P := \mathrm{conv}(\sigma^{\mathrm{k}} \cup_{\{0\}} \sigma^{\mathrm{l}})$ where σ^k and σ^l are embedded in orthogonal spaces and intersect only in the origin which is in the relative interior of both. McMullen's proof of the g-theorem for simplicial polytopes [47, 46] states that $\sum_{v\in P_0} x_v = \omega_{\partial\sigma^k} + \omega_{\partial\sigma^l}$ is indeed an HL element of $\mathbb{R}[\partial\sigma^k * \partial\sigma^l]/(\Theta_{\partial P})$ where $\Theta_{\partial P}$ is the l.s.o.p. induced by the positions of the vertices in the polytope P. By the definition of P, $\Theta_{\partial P} = \Theta_{\partial\sigma^k} \uplus \Theta_{\partial\sigma^l}$. Thus (2) is proved. $\square$

Remark: As a nonzero multiple of an HL element is again HL, then in Theorem 4.2.4(2) any element $a\omega_K + b\omega_L$ where $a, b \in \mathbb{R}$, $ab \neq 0$, will do.

Corollary 4.2.5 *Let K and L be HL simplicial/homology/piecewise linear spheres of dimensions k, l respectively. Then their join $K * L$ is an HL $(k+l+1)$-simplicial/homology/piecewise linear sphere.*

Proof : As simplicial/homology/piecewise linear spheres are Gorenstein*, the corollary follows at once from Theorem 4.2.4 and the fact that join of simplicial/homology/piecewise linear spheres is again a simplicial/homology/piecewise linear sphere of appropriate dimension. $\square$

4.3 Weak Lefschetz and gluing

The proof of the following proposition is similar to the proof that pure shellable complexes are Cohen-Macaulay due to Stanley [66]; see also [12], Theorem 5.1.13.

Proposition 4.3.1 *(with Yhonatan Iron) Let K, L and $K \cap L$ be simplicial complexes of the same dimension $d-1$. Assume that K and L are weak-Lefschetz. If $K \cap L$ is CM then $K \cup L$ is weak-Lefschetz.*

Proof : For any two complexes K, L with $(K \cup L)_0 = [n]$ the inclusions $K \cap L \subseteq K, L \subseteq K \cup L$ induce a short exact sequence of $A = \mathbb{R}[x_1, ..., x_n]$

modules

$$0 \longrightarrow \mathbb{R}[K \cup L] \longrightarrow \mathbb{R}[K] \oplus \mathbb{R}[L] \longrightarrow \mathbb{R}[K \cap L] \longrightarrow 0 \quad (4.7)$$

(maps are given by projections). The above four complexes have the same dimension, hence they have a common l.s.o.p. Θ (as intersection of finitely many nonempty Zariski open sets is nonempty). As the functor $\otimes_A A/\Theta$ is right exact, we obtain the following commutative diagram of exact sequences for each $1 \leq i \leq \lfloor d/2 \rfloor$:

$$\begin{tikzcd}
\mathrm{Tor}(\mathbb{R}[\mathrm{K}\cap\mathrm{L}], \frac{\mathrm{A}}{\Theta})_{\mathrm{i}-1} \arrow[r, "\delta_{i-1}"] \arrow[d] & \frac{\mathbb{R}[K\cup L]}{(\Theta)}_{i-1} \arrow[r] \arrow[d, "\omega"] & \frac{\mathbb{R}[K]}{(\Theta)}_{i-1} \oplus \frac{\mathbb{R}[L]}{(\Theta)}_{i-1} \arrow[r] \arrow[d, "(\omega,\omega)"] & \frac{\mathbb{R}[K\cap L]}{(\Theta)}_{i-1} \arrow[r] \arrow[d, "\omega"] & 0 \arrow[d] \\
\mathrm{Tor}(\mathbb{R}[\mathrm{K}\cap\mathrm{L}], \frac{\mathrm{A}}{\Theta})_{\mathrm{i}} \arrow[r, "\delta_i"] & \frac{\mathbb{R}[K\cup L]}{(\Theta)}_{i} \arrow[r] & \frac{\mathbb{R}[K]}{(\Theta)}_{i} \oplus \frac{\mathbb{R}[L]}{(\Theta)}_{i} \arrow[r] & \frac{\mathbb{R}[K\cap L]}{(\Theta)}_{i} \arrow[r] & 0
\end{tikzcd} \quad (4.8)$$

where the horizontal arrows preserve grading and the vertical arrows are multiplication by a generic $\omega \in A_1$, i.e. ω is a WL element for both K and L. (For the middle terms we used distributivity of $\otimes$ and $\oplus$.)

In order to show that $\omega : (\frac{\mathbb{R}[K\cup L]}{(\Theta)})_{i-1} \to (\frac{\mathbb{R}[K\cup L]}{(\Theta)})_i$ is injective, it is enough to show that $\delta_{i-1} = 0$, which of course holds if $\mathrm{Tor}(\mathbb{R}[\mathrm{K}\cap\mathrm{L}], \mathrm{A}/\Theta)_{\mathrm{i}-1} = 0$. Note that for an A-module M $\mathrm{Tor}(\mathrm{M}, \mathrm{A}/\Theta) = \mathrm{Ker}(\mathrm{M}\otimes_{\mathrm{A}}(\Theta) \to \mathrm{M})$. As $M = \mathbb{R}[K \cap L]$ is CM, it is a free $\mathbb{R}[\Theta]$-module, hence $\mathrm{Tor}(\mathbb{R}[\mathrm{K}\cap\mathrm{L}], \mathrm{A}/\Theta)_{\mathrm{i}-1} = 0$ for every i. $\square$

Remarks: (1) Note that the above proof provides an even more general condition on $K \cap L$ which already guarantees that $K \cup L$ is WL.

(2) Compare Proposition 4.3.1 to [72], remark after proof of Theorem 3.9: there the gluing corresponds to an ear decomposition of homology spheres and balls.

4.4 Lefschetz properties and connected sum

Let K and L be pure simplicial complexes which intersect in a common facet $< \sigma >= K \cap L$. Their *connected sum over* σ is $K\#_\sigma L = (K \cup L) \setminus \{\sigma\}$.

Theorem 4.4.1 *Let K and L be Gorenstein* complexes over $\mathbb{F}$ which intersect in a common facet $< \sigma >= K \cap L$, of dimension $d-1$. Let $A = \mathbb{F}[x_v : v \in (K \cup L)_0]$. Then:*

(0) $K\#_\sigma L$ is Gorenstein* of dimension d; in particular its h-vector is symmetric.

(1) Let Θ be a common l.s.o.p for K, L, $< \sigma >$ and $K\#_\sigma L$ over A (it exists) and assume that ω is an HL element for both K and L w.r.t. Θ. Then ω is an i-Lefschetz element of $\mathbb{F}[K\#_\sigma L]/(\Theta)$ for $0 < i \leq d/2$.

(2) $K\#_\sigma L$ is HL.

Proof: A straightforward Mayer-Vietoris and Euler characteristic argument shows that $K\#_\sigma L$ is Gorenstein*, and hence has a symmetric h-vector. It is also easy to compute directly that $h(K\#_\sigma L) = h(K) + h(L) - (1, 0, 0, ..., 0, 1)$, a sum of symmetric vectors, and hence is symmetric; also $h_0 = h_d = 1$.

For a simplicial complex L let $\mathbb{F}(L) := \bigoplus_{a:\mathrm{supp(a)}\in \mathrm{L}} \mathbb{F}x^a$ be an $A_{L_0} = \mathbb{F}[x_v : v \in L_0]$ module defined by $x_v(x^a) = \begin{cases} x_v x^a & \text{if } \mathrm{v}\cup\mathrm{supp(a)}\in\mathrm{L} \\ 0 & \text{otherwise} \end{cases}$. Note that $\mathbb{F}(L) \cong \mathbb{F}[L]$ as A_{L_0}-modules.

Then the following is an exact sequence of A-modules:

$$0 \to \mathbb{F}(< \sigma >) \xrightarrow{(\iota, -\iota)} (\mathbb{F}(K) \oplus \mathbb{F}(L)) \xrightarrow{\iota_K + \iota_L} \mathbb{F}(K \cup_\sigma L) \to 0 \tag{4.9}$$

where the ι's denote the obvious inclusions. As a finite intersection of Zariski nonempty open sets is nonempty, Θ as in (1) exists (see Lemma 4.5.1). When we mod out Θ from (4.9), which is the same as tensor (4.9) with $\otimes_A A/\Theta$, we obtain an exact sequence:

$$(\mathbb{F}(< \sigma >)/(\Theta)) \to (\mathbb{F}(K)/(\Theta) \oplus \mathbb{F}(L)/(\Theta)) \to (\mathbb{F}(K \cup_\sigma L)/(\Theta)) \to 0 \tag{4.10}$$

where in the middle term we used distributivity of $\otimes$ and $\oplus$. Note that $\mathbb{F}(< \sigma >)/(\Theta) \cong \mathbb{F}$ is concentrated in degree 0 and that $(\mathbb{F}(K\#_\sigma L)/(\Theta))_{<d} \cong (\mathbb{F}(K \cup_\sigma L)/(\Theta))_{<d}$. Thus, for $0 < i \leq d/2$ we obtain the following commutative diagram:

$$\begin{array}{ccccc} (\frac{\mathbb{F}(K\#_\sigma L)}{(\Theta)})_i & \xrightarrow{\cong} & (\frac{\mathbb{F}(K\cup_\sigma L)}{(\Theta)})_i & \xrightarrow{\cong} & (\frac{\mathbb{F}(K)}{(\Theta)})_i \bigoplus (\frac{\mathbb{F}(L)}{(\Theta)})_i \\ \downarrow{\scriptstyle \omega^{d-2i}} & & \downarrow{\scriptstyle \omega^{d-2i}} & & \downarrow{\scriptstyle \omega^{d-2i} \oplus \omega^{d-2i}} \\ (\frac{\mathbb{F}(K\#_\sigma L)}{(\Theta)})_{d-i} & \xrightarrow{\cong} & (\frac{\mathbb{F}(K\cup_\sigma L)}{(\Theta)})_{d-i} & \xrightarrow{\cong} & (\frac{\mathbb{F}(K)}{(\Theta)})_{d-i} \bigoplus (\frac{\mathbb{F}(L)}{(\Theta)})_{d-i} \end{array} \tag{4.11}$$

where the right vertical arrow is an isomorphism by assumption. Hence, the left vertical arrow is an isomorphism as well, meaning that ω is an i-Lefschetz element of $\mathbb{F}[K\#_\sigma L]/(\Theta)$ for $0 < i \leq d/2$.

For $i = 0$, as $K\#L$ is Cohen-Macaulay with l.s.o.p. Θ and $h_d = 1$, then there exists a 0-Lefschetz element $\tilde{\omega}$ (i.e. $\tilde{\omega}^d \neq 0$. This is equivalent to $[2, d+1] \in \Delta^s(K\#L)$, which reflects the fact that $K\#L$ has non-vanishing top homology.). By Lemma 4.5.1 the sets of 0-Lefschetz elements and of $(0 <)$-Lefschetz elements are Zariski open. The fact that they are nonempty

implies that so is their intersection, i.e. $K\#L$ is HL. $\square$
Remark: (2) follows also from the symmetric case of Corollary 2.1.7. The proof given here in our 'special case' is simpler.

Corollary 4.4.2 *Let K and L be HL simplicial spheres of the same dimension d. Then their connected sum $K\#L$ is also an HL d-sphere.* $\square$

Corollary 4.4.3 *Let K and L be WL spheres of the same dimension, which intersect in a common facet $<\sigma>= K\cap L$. Then $K\#_{\sigma}L$ is WL.*

Proof: Imitate the proof of Theorem 4.4.1. $\square$

4.5 Swartz lifting theorem and beyond

In this section we show that for proving Conjecture 4.1.1 it suffices to show that $y_{d+1} : H(K,\Theta)_{\lfloor d/2\rfloor} \to H(K,\Theta)_{\lceil d/2\rceil}$ is an isomorphism for K a homology $(d-1)$-sphere with d odd and generic l.s.o.p. Θ and y_{d+1} in A_1. We end this section by stating a stronger conjecture about the structure of the set of pairs (Θ,ω) of an l.s.o.p. and a $\lfloor d/2\rfloor$-Lefschetz element (stronger than being nonempty), which hopefully would be easier to prove.

Consider the multiplication maps $\omega_i : H(K,\Theta)_i \longrightarrow H(K,\Theta)_{i+1}$, $m \mapsto \omega_i m$ where $\omega_i \in A_1$. Let $\dim(K) = d-1$. Denote by $\Omega_{UWL}(K,i)$ the set of all $(\Theta,\omega_i) \in A_1^{\dim(K)+2}$ such that Θ is an l.s.o.p. of $\mathbb{R}[K]$, $\mathbb{R}[K]$ is a free $\mathbb{R}[\Theta]$-module, and $\omega_i : H(K)_i \longrightarrow H(K)_{i+1}$ is injective for $i < d/2$ and surjective for $i \geq d/2$. Denote by $\Omega_{HL}(K,i)$ the set of all $(\Theta,\omega) \in (A_{K_0})_1^{d+1}$ such that Θ is an l.s.o.p. of $\mathbb{R}[K]$, $\mathbb{R}[K]$ is a free $\mathbb{R}[\Theta]$-module, and $\omega^{d-2i} : H(K)_i \longrightarrow H(K)_{d-i}$ is injective $(0 \leq i \leq \lfloor d/2\rfloor)$. For d odd $\Omega_{UWL}(K,\lfloor d/2\rfloor) = \Omega_{HL}(K,\lfloor d/2\rfloor)$, which we simply denote by $\Omega(K,\lfloor d/2\rfloor)$.

The following was proved by Swartz [72], Proposition 3.6 for $\Omega_{HL}(K,i)$; similar arguments can be used to prove the same conclusion for $\Omega_{UWL}(K,i)$.

Lemma 4.5.1 *(Swartz) For every simplicial complex K and for every i, $\Omega_{UWL}(K,i)$ is a Zariski open set. For $0 \leq i \leq \lfloor \frac{\dim(K)+1}{2}\rfloor$, $\Omega_{HL}(K,i)$ is a Zariski open set. (They may be empty, e.g. if K is not pure.)*

Theorem 4.5.2 *(Swartz) Let $d \geq 1$. If for every homology $2d$-sphere L, $\Omega(L,d)$ is nonempty, then for every $t > 2d$ and for every homology t-sphere K, $\Omega_{UWL}(K,m)$ is nonempty for every $m \leq d$.*

Proof: By [71], Theorem 4.26 and induction on t, $\Omega_{UWL}(K,(t+1)-(d+1))$ is nonempty, i.e. multiplication $\omega : H(K)_{t-d} \to H(K)_{t-d+1}$ is surjective for a generic $\omega \in A_1$. As the ring $H(K)$ is standard, $\Omega_{UWL}(K,(t+1)-(m+1))$ is nonempty for every $m \leq d$. Hence, for the canonical module $\Omega(K)$, multiplication by a generic degree 1 element $\omega : (\Omega(K)/\Theta\Omega(K))_m \to (\Omega(K)/\Theta\Omega(K))_{m+1}$ is injective in the first d degrees. As K is a homology sphere, $\Omega(K) \cong \mathbb{R}[K]$, hence $\Omega_{UWL}(K,m)$ is nonempty for every $m \leq d$. $\square$

For more information about canonical modules we refer to [69].

Combined with Lemma 4.5.1, and the fact that a finite intersection of Zariski nonempty open sets is nonempty, if the conditions of Theorem 4.5.2 are met for every $d \geq 1$ then every homology sphere is unimodal WL, and hence Conjecture 4.1.1 follows.

We wish to show further, that if 'all' even dimensional spheres satisfy the condition in Theorem 4.5.2 then 'all' spheres are HL. By 'all' we mean a family of Gorenstein* simplicial complexes which contains all boundaries of simplices and which is closed under joins and links (e.g. homology /simplicial /PL spheres). The following lemma provides a step in this direction.

Lemma 4.5.3 *Let S be a Gorenstein* simplicial complex with an l.s.o.p. Θ_S over $\mathbb{R}$. If $H(S,\Theta_S)$ is $(\lfloor\frac{\dim S+1}{2}\rfloor)$-Lefschetz but not HL then there exists a simplex σ such that $S * \partial\sigma$ is of even dimension $2j$, and for every l.s.o.p. $\Theta_{\partial\sigma}$ of $\partial\sigma$, $\mathbb{R}[S*\partial\sigma]/(\Theta_S \cup \Theta_{\partial\sigma})$ has no j-Lefschetz element; in particular $S*\partial\sigma$ is not unimodal WL. (We would like to obtain this conclusion for* every *l.s.o.p. of $S*\partial\sigma$!)*

Proof : Denote the dimension of S by $d-1$ and recall that $A_{S_0} = \mathbb{R}[x_v : v \in S_0]$. By Lemma 4.5.1 $\Omega_{HL}(S,i)$ is a Zariski open set for every $0 \leq i \leq \lfloor d/2 \rfloor$. The assumption that S is not HL (but is $(\lfloor\frac{d}{2}\rfloor)$-Lefschetz) implies that there exists $0 \leq i_0 \leq \lfloor d/2\rfloor - 1$ such that $\Omega_{HL}(S,i_0) = \emptyset$ (as a finite intersection of Zariski nonempty open sets is nonempty). Hence, for the fixed l.s.o.p. Θ_S and every $\omega_S \in (A_{S_0})_1$, there exists $0 \neq m = m(\omega_S) \in H_{i_0}(S)$ such that $\omega_S^{d-2i_0} m = 0$.

Let $T = S*\partial\sigma$ where σ is the $(d-2i_0-1)$-simplex. Note that $\dim(\sigma) \geq 1$ (as S is $(\lfloor\frac{\dim S+1}{2}\rfloor)$-Lefschetz), hence $\partial\sigma \neq \emptyset$. Then T is of even dimension $2d-2i_0-2$. We have seen (Theorem 4.2.4) that for any l.s.o.p. $\Theta_{\partial\sigma}$ of $\partial\sigma$, $\Theta_T := \Theta_S \cup \Theta_{\partial\sigma}$ is an l.s.o.p. of T. Every $\omega_T \in (A_{T_0})_1$ has a unique expansion $\omega_T = \omega_S + \omega_{\partial\sigma}$ where $\omega_S \in (A_{S_0})_1$ and $\omega_{\partial\sigma} \in (A_{\partial\sigma_0})_1$. Recall the isomorphism (4.6) of A_{T_0}-modules $\mathbb{R}[T]/(\Theta_T) \cong \mathbb{R}[S]/(\Theta_S) \otimes_{\mathbb{R}} \mathbb{R}[\partial\sigma]/(\Theta_{\partial\sigma})$. Let $m(\omega_T) \in (\frac{\mathbb{R}[T]}{(\Theta_T)})_{d-i_0-1}$ be

$$m(\omega_T) := \sum_{0\leq j\leq d-2i_0-1} (-1)^j \omega_S^{d-2i_0-1-j} m \otimes \omega_{\partial\sigma}^j 1.$$

Note that the sum $\omega_T m(\omega_T)$ is telescopic, thus $\omega_T m(\omega_T) = \omega_S^{d-2i_0} m \otimes 1 + (-1)^{d-2i_0-1} m \otimes \omega_{\partial\sigma}^{d-2i_0} 1 = 0 + 0 = 0$. For a generic ω_T, the projection of $\omega_{\partial\sigma}$ on $\mathbb{R}[\partial\sigma]/(\Theta_{\partial\sigma})$ is nonzero, hence so is the projection of $\omega_{\partial\sigma}^{d-2i_0-1}$, and we get that $m(\omega_T) \neq 0$. Thus, Zariski topology tells us that for *every* $\omega_T \in (A_{T_0})_1$, there exists $0 \neq m(\omega_T) \in (\frac{\mathbb{R}[T]}{(\Theta_T)})_{d-i_0-1}$ such that $\omega_T m(\omega_T) = 0$. $\square$

We conjecture that the following stronger property holds for $\Omega(L, d)$:

Conjecture 4.5.4 *Let L be a homology $2d$-sphere ($d \geq 1$) on n vertices. Then $\Omega(L, d)$ intersects every hyperplane in the vector space $A_1^{2d+2} \cong \mathbb{R}^{(2d+2)n}$.*

For L the boundary of a simplex, the complement of $\Omega(L, d)$ is the set of all $(z_1, ..., z_{2d+2}) \in A_1^{2d+2}$ such that $\det(z_1, ..., z_{2d+2}) = 0$. As $\det \in \mathbb{R}[z_{i,j}]_{1 \leq i,j \leq 2d+2}$ is an irreducible polynomial, in particular it has no linear factor, hence $\Omega(L, d)$ intersects every hyperplane. By an unpublished argument of Swartz, it follows that if L' is obtained from L by a bistellar move, and $\Omega(L, d)$ intersects every hyperplane then $\Omega(L', d)$ is nonempty. We need to show that $\Omega(L', d)$ intersects every hyperplane, in order to conclude that the g-conjecture holds for PL-spheres. 'Unfortunately', $\Omega(L, d)$ may not be connected, as its complement is a codimension one algebraic variety.

4.6 Lefschetz properties and Stellar subdivisions

Roughly speaking, we will show that Stellar subdivisions preserve the HL property.

Proposition 4.6.1 *Let K be a simplicial complex. Let K' be obtained from K by identifying two distinct vertices u and v in K, i.e. $K' = \{T : u \notin T \in K\} \cup \{(T \setminus \{u\}) \cup \{v\} : u \in T \in K\}$. Let $d \geq 2$. Assume that $\{d+2, d+3, ..., 2d+1\} \notin \Delta(K')$ and that $\{d+1, d+2, ..., 2d-1\} \notin \Delta(\mathrm{lk}(u, K) \cap \mathrm{lk}(v, K))$. Then $\{d+2, d+3, ..., 2d+1\} \notin \Delta(K)$. (Shifting is over $\mathbb{R}$.)*

Remark: The case $d = 2$ and $\dim(K) = 1$ follows from Lemmata 3.1.2 (symmetric case) and 3.3.1 (exterior case).

Proof for symmetric shifting: (with Eric Babson) Let $\psi : K_0 \longrightarrow \mathbb{R}^{2d}$ be a generic embedding, i.e. all minors of the representing matrix w.r.t. a fixed basis are nonzero. It induces the following map:

$$\psi_K^{2d} : \oplus_{T \in K_{d-1}} \mathbb{R}T \longrightarrow \oplus_{F \in \binom{K_0}{d-1}} \mathbb{R}^{2d} / \mathrm{span}(\psi(F)),$$

$$1T \mapsto \sum_{F \in \binom{K_0}{d-1}} \delta_{F \subseteq T} \overline{\psi(T \setminus F)} F \qquad (4.12)$$

where $\delta_{F\subseteq T}$ equals 1 if $F \subseteq T$ and 0 otherwise.

Recall that $\{d+2, d+3, ..., 2d+1\} \notin \Delta^s(K)$ iff $y^d_{2d+1} \notin GIN(K)$, where $Y = \{y_i\}_i$ is a generic basis for A_1, $A = \mathbb{R}[x_v : v \in K_0]$. By Lee [39] Theorems 10,12,15 and Tay, White and Whiteley [73] Proposition 5.2, $y^d_{2d+1} \notin GIN(K)$ iff $\operatorname{Ker} \phi^{2d}_K = 0$ for some $\phi : K_0 \longrightarrow \mathbb{R}^{2d}$ (equivalently, every ϕ in some Zariski non-empty open set of embeddings).

Consider the following degenerating map: for $0 < t \leq 1$ let $\psi_t : K_0 \longrightarrow \mathbb{R}^{2d}$ be defined by $\psi_t(i) = \psi(i)$ for every $i \neq u$ and $\psi_t(u) = \psi(v) + t(\psi(u) - \psi(v))$. Thus $\psi_1 = \psi$, and $\lim_{t\mapsto 0}(\psi_t(u) - \psi_t(v)) = \psi(u) - \psi(v)$. Let $\psi_0 = \lim_{t\mapsto 0} \psi_t$.

Let $\psi^{2d}_{K,t} : \oplus_{T\in K_{d-1}} \mathbb{R}T \longrightarrow \oplus_{F\in\binom{K_0}{d-1}} \mathbb{R}^{2d}/\operatorname{span}(\psi_t(F))$ be the map induced by ψ_t; thus $\psi^{2d}_{K,1} = \psi^{2d}_K$. Denote $\psi^{2d}_0 = \lim_{t\mapsto 0} \psi^{2d}_{K,t}$. Assume for a moment that ψ^{2d}_0 is injective. Then for a small enough perturbation of the entries of a representing matrix of ψ^{2d}_0, the columns of the resulted matrix would be independent, i.e. the corresponding linear transformation would be injective. In particular, there would exist an $\epsilon > 0$ such that for every $0 < t < \epsilon$, $\operatorname{Ker} \psi^{2d}_{K,t} = 0$, and hence for every $\phi : K_0 \longrightarrow \mathbb{R}^{2d}$ in some Zariski non-empty open set of embeddings, $\operatorname{Ker} \phi^{2d}_K = 0$. Thus, the following Lemma 4.6.2 completes the proof. $\square$

Lemma 4.6.2 *ψ^{2d}_0 is injective for a non-empty Zariski open set of embeddings $\psi : K_0 \longrightarrow \mathbb{R}^{2d}$.*

Proof : For every $0 < t \leq 1$ and every F such that $\{u, v\} \subseteq F \in \binom{K_0}{d-1}$, $\operatorname{span}(\psi_t(F)) = \operatorname{span}(\psi(F))$, and hence in the range of ψ^{2d}_0 we mod out by $\operatorname{span}(\psi(F))$ for summands with such F. For summands of $\{u, v\} \nsubseteq F \in \binom{K_0}{d-1}$, we mod out by $\operatorname{span}(\psi_0(F))$. Note that for T such that $\{u, v\} \subseteq T \in K_{d-1}$,

$$\psi^{2d}_0(T)|_{T\setminus v} = (\psi(u) - \psi(v)) + \operatorname{span}(\psi(T \setminus u)) = -\psi^{2d}_0(T)|_{T\setminus u}.$$

For a linear transformation C, denote by $[C]$ its representing matrix w.r.t. given bases. In $[\psi^{2d}_0]$ bases are indexed by sets as in (4.12). First add rows $F' \uplus \{u\}$ to rows $F' \uplus \{v\}$, then delete the rows F containing u, to obtain a matrix $[B]$, of a linear transformation B. In particular, we delete all rows F such that $\{u, v\} \subseteq F$.

Note that $K'_0 = K_0 \setminus \{u\}$, thus, for the obvious bases, $[B]$ is obtained from $[(\psi|_{K'_0})^{2d}_{K'}]$ by doubling the columns indexed by $T' \uplus \{v\} \in K'_{d-1}$ where both $T' \uplus \{v\}, T' \uplus \{u\} \in K_{d-1}$, and by adding a zero column for every $T' \uplus \{u, v\} \in K_{d-1}$. For short, denote $\psi^{2d}_{K'} = (\psi|_{K'_0})^{2d}_{K'}$. More precisely, the linear maps B and $\psi^{2d}_{K'}$ are related as follows: they have the same range. The

domain of B is $\mathrm{dom}(B) = \mathrm{dom}(\psi_0^{2d}) = D_1 \oplus D_2 \oplus D_3$ where
$D_1 = \oplus\{\mathbb{R}T : T \in K_{d-1}, \{u,v\} \nsubseteq T, (u \in T) \Rightarrow (T \setminus u) \cup v \notin K\}$,
$D_2 = \oplus\{\mathbb{R}T : T \in K_{d-1}, u \in T, v \notin T, (T \setminus u) \cup v \in K\}$,
$D_3 = \oplus\{\mathbb{R}T : T \in K_{d-1}, \{u,v\} \subseteq T\}$.
For a base element $1T$ of D_1, let $T' \in K'$ be obtained from T by replacing u with v. Then $B(1T) = \psi_{K'}^{2d}(1T')$; thus $\mathrm{Ker}\, B|_{D_1} \cong \mathrm{Ker}\, \psi_{K'}^{2d}$. For a base element $1T$ of D_2, $B(1T) = \psi_{K'}^{2d}(1((T \setminus u) \cup v))$, and $B|_{D_3} = 0$.

Assume we have a linear dependency $\sum_{T \in K_{d-1}} \alpha_T \psi_0^{2d}(T) = 0$. By assumption, $\{d+2, d+3, ..., 2d+1\} \notin \Delta^s(K')$, hence $\mathrm{Ker}\, \psi_{K'}^{2d} = 0$, thus $\alpha_T = 0$ for every base element T except possibly for $T \in D_3$ and for $T' \uplus \{u\}, T' \uplus \{v\} \in K_{d-1}$, where $\alpha_{T' \uplus \{u\}} = -\alpha_{T' \uplus \{v\}}$.

Let $\psi_0^{2d}|_{\mathrm{res}}$ be the restriction of ψ_0^{2d} to the subspace spanned by the base elements T such that $v \in T$ and for which it is (yet) not known that $\alpha_T = 0$, followed by projection into the subspace spanned by the $F \in \binom{K_0}{d-1}$ coordinates where $v \in F$ - just forget the other coordinates. As $\psi_0^{2d}(T)|_F = 0$ whenever $F \ni v \notin T$, if $\psi_0^{2d}|_{\mathrm{res}}$ is injective, then $\alpha_T = 0$ for all $T \in K_{d-1}$. Thus, the Lemma 4.6.3 below completes the proof. $\square$

Lemma 4.6.3 $\psi_0^{2d}|_{\mathrm{res}}$ *is injective for a non-empty Zariski open set of embeddings* $\psi : K_0 \longrightarrow \mathbb{R}^{2d}$.

Proof : Let $G = (\{u\} * (\mathrm{lk}(u,K) \cap \mathrm{lk}(v,K)))_{\leq d-2}$. Note that v appears in the index set of every row and every column of $[\psi_0^{2d}|_{\mathrm{res}}]$. Omitting v from the indices of both of the bases used to define $\psi_0^{2d}|_{\mathrm{res}}$, we notice that

$$\psi_0^{2d}|_{\mathrm{res}} \cong \overline{\psi_0^{2d}|_{\mathrm{res}}} : \oplus_{T \in G_{d-2}} \mathbb{R}T \longrightarrow \oplus_{F \in \binom{G_0}{d-2}} \mathbb{R}^{2d} / \mathrm{span}(\psi(F \uplus \{v\})) =$$

$$\oplus_{F \in \binom{G_0}{d-2}} (\mathbb{R}^{2d} / \mathrm{span}(\psi(v))) / \overline{span(\psi(F))},$$

$$1T \mapsto \sum_{F \in \binom{G_0}{d-2}} \delta_{F \subseteq T} \overline{\psi(T \setminus F)} F$$

where $\delta_{F \subseteq T}$ equals 1 if $F \subseteq T$ and 0 otherwise, and $\overline{\mathrm{span}(\psi(F))}$ is the image of $\mathrm{span}(\psi(F))$ in the quotient space $\mathbb{R}^{2d} / \mathrm{span}(\psi(v))$.

Consider the projection $\pi : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d} / \mathrm{span}(\psi(v)) \cong \mathbb{R}^{2d-1}$. Let $\bar{\psi} = \pi \circ \psi|_{G_0} : G_0 \longrightarrow \mathbb{R}^{2d-1}$, and $\bar{\psi}_G^{2d-1}$ be the induced map as defined in (4.12). Then π induces $\pi_* \overline{\psi_0^{2d}|_{\mathrm{res}}} = \bar{\psi}_G^{2d-1}$.

By assumption, $\{d+1, ..., 2d-1\} \notin \Delta^s(\mathrm{lk}(u,K) \cap \mathrm{lk}(v,K))$. As symmetric shifting commutes with constructing a cone (Kalai [34] Theorem 2.2.8, and Babson, Novik and Thomas [3] Theorem 3.7), $\{d+2, ..., 2d\} \notin \Delta^s(G)$. Hence $y_{2d}^{d-1} \notin \mathrm{GIN}(G)$, and by Lee [39], $\mathrm{Ker}\, \phi_G^{2d-1} = 0$ for a generic ϕ. Thus, all

liftings $\psi : K_0 \longrightarrow \mathbb{R}^{2d}$ such that $\bar{\psi} = \phi$ satisfy $\operatorname{Ker} \psi_0^{2d}|_{\text{res}} \cong \operatorname{Ker} \phi_G^{2d-1} = 0$, and this set of liftings is a non-empty Zariski open set. $\square$

Remark: Clearly the set of all ψ such that ψ_K^{2d} is injective is Zariski open. We exhibited conditions under which it is non-empty.

Proof for exterior shifting: The proof is similar to the proof for the symmetric case. We indicate the differences. $\psi : K_0 \to \mathbb{R}^{d+1}$ defines the first $d+1$ generic f_i's w.r.t. the e_i's basis of $\mathbb{R}^{|K_0|}$ and induces the following map:

$$\psi_{K,\text{ext}}^{d+1} : \oplus_{T \in K_{d-1}} \mathbb{R}T \longrightarrow \oplus_{1 \leq i \leq d+1} \oplus_{F \in \binom{K_0}{d-1}} \mathbb{R}F, \; m \mapsto (f_1 \lfloor m, ..., f_{d+1} \lfloor m) \tag{4.13}$$

By Proposition 1.2.1, $\operatorname{Ker} \psi_{K,\text{ext}}^{d+1} = \cap_{1 \leq i \leq d+1} \operatorname{Ker}_{d-1} f_i \lfloor = \cap_{R < \{d+2,...,2d+1\}} \operatorname{Ker}_{d-1} f_R \lfloor$, hence, by shiftedness, $\{d+2, ..., 2d+1\} \notin \Delta^e(K) \Leftrightarrow \operatorname{Ker} \psi_{K,\text{ext}}^{d+1} = 0$.

Replacing $\psi(u)$ by $\psi(v)$ induces a map

$$\psi_{K,u}^{d+1} : \oplus_{T \in K_{d-1}} \mathbb{R}T \longrightarrow \oplus_{1 \leq i \leq d+1} \oplus_{F \in \binom{K_0}{d-1}} \mathbb{R}F.$$

By perturbation, if $\operatorname{Ker} \psi_{K,u}^{d+1} = 0$ then $\operatorname{Ker} \psi_{K,\text{ext}}^{d+1} = 0$ for generic ψ.

Let $[B_{\text{ext}}]$ be obtained from the matrix $[\psi_{K,u}^{d+1}]$ by adding the rows $F' \uplus u$ to the corresponding rows $F' \uplus v$ and deleting the rows F with $\{u, v\} \subseteq F$. The domain of B_{ext} is $D_1 \oplus D_2 \oplus D_3$ as for B in the symmetric case. For a base element $1T$ of D_1, let $T' \in K'$ be obtained from T by replacing u with v. Then $B_{\text{ext}}(1T) = \psi_{K',\text{ext}}^{d+1}(1T')$; thus $\operatorname{Ker} B_{\text{ext}}|_{D_1} \cong \operatorname{Ker} \psi_{K',\text{ext}}^{d+1}$. For a base element $1T$ of D_2, $B_{\text{ext}}(1T) = \psi_{K',\text{ext}}^{d+1}(1((T \setminus u) \cup v))$, and as we may number $v = 1, u = 2$ then $B|_{D_3} = 0$ (the rows of $F' \uplus u$ and of $F' \uplus v$ have opposite sign in $\psi_{K,u}^{d+1}$). Now we can repeat the arguments showing that $\operatorname{Ker} \psi_0^{2d} = 0$ by considering B in the symmetric case, to show that $\operatorname{Ker} \psi_{K,u}^{d+1} = 0$ by considering B_{ext}. $\square$

Corollary 4.6.4 *Let K be a $2d$-sphere for some $d \geq 1$, and let $a, b \in K$ be two vertices which satisfy the* Link Condition, *i.e that* $\mathrm{lk(a, K)} \cap \mathrm{lk(b, K)} = \mathrm{lk(\{a, b\}, K)}$. *Let K' be obtained from K by contracting $a \mapsto b$. Then:*

(1) K' is a $2d$-sphere, PL homeomorphic to K (see Theorem 5.4.1).

(2) If K' is d-Lefschetz and $\mathrm{lk}(\{a, b\}, K)$ *is $(d-1)$-Lefschetz, then K is d-Lefschetz (by Proposition 4.6.1).* $\square$

Let K be a simplicial complex. Its *Stellar subdivision at a face $T \in K$* is the operation $K \mapsto K'$ where $K' = \mathrm{Stellar}(T, K) := (K \setminus \mathrm{st}(T, K)) \cup (\{v_T\} * \partial T * \mathrm{lk}(T, K))$, where v_T is a vertex not in K. Note that for $u \in T \in K$, $u, v_T \in K'$ satisfy the Link Condition and their identification results in K. Further, $\mathrm{lk}(\{u, v_T\}, K') = \mathrm{lk}(u, \partial T * \mathrm{lk}(T, K)) = \partial(T \setminus u) * \mathrm{lk}(T, K)$.

Theorem 4.6.5 *Let S be a homology sphere and $F \in S$. If S and* $\mathrm{lk}(F, S)$ *are HL then* $\mathrm{Stellar}(F, S)$ *is HL.*

Proof : Let $T = \mathrm{Stellar}(F, S)$, denote its dimension by $d-1$, and assume by contradiction that T is not HL. As we have seen in the proof of Lemma 4.5.3, there exists $0 \leq i_0 \leq \lfloor d/2 \rfloor$ such that $\Omega_{HL}(T, i_0) = \emptyset$. First we show that $i_0 \neq \lfloor d/2 \rfloor$: for d even this is obvious. For d odd, note that for $u \in F$ the contraction $v_F \mapsto u$ in T results in S, which is $\lfloor d/2 \rfloor$-Lefschetz. Further, the $(d-3)$-sphere $\mathrm{lk}(\{v_F, u\}, T) = \mathrm{lk}(F, S) * \partial(F \setminus \{u\})$ is HL by Theorem 4.2.4, and in particular is $(\lfloor d/2 \rfloor - 1)$-Lefschetz. Thus, by Corollary 4.6.4 T is $\lfloor d/2 \rfloor$-Lefschetz, and hence $0 \leq i_0 \leq \lfloor d/2 \rfloor - 1$.

Let $L = T * \partial\sigma$, where σ is the $(d - 2i_0 - 1)$-simplex (then L has even dimension $2d - 2i_0 - 2$). By Lemma 4.5.3, for any two l.s.o.p.'s Θ_T and $\Theta_{\partial\sigma}$ of $\mathbb{R}[T]$ and $\mathbb{R}[\partial\sigma]$ respectively, $\mathbb{R}[L]/(\Theta_T \cup \Theta_{\partial\sigma})$ has no $(d-i_0-1)$-Lefschetz element.

On the other hand, we shall now prove the existence of such l.s.o.p.'s and a $(d - i_0 - 1)$-Lefschetz element, to reach a contradiction. This requires a close look on the proof of Proposition 4.6.1.

Note that $L = \mathrm{Stellar}(F, S*\partial\sigma)$, and that for $u \in F$ the contraction $v_F \mapsto u$ in L results in $S * \partial\sigma$. Further, $\mathrm{lk}(\{v_F, u\}, L) = \mathrm{lk}(F, S) * \partial(F \setminus \{u\}) * \partial\sigma$.

Applying Zariski topology considerations to subspaces of the space of embeddings $\{f : L_0 \to \mathbb{R}^{2d-2i_0}\} \cong \mathbb{R}^{|L_0| \times (2d-2i_0)}$, we now show that there exists an embedding $\psi : L_0 \longrightarrow \mathbb{R}^d \oplus \mathbb{R}^{d-2i_0-1} \oplus \mathbb{R}$ such that the following three properties hold *simultaneously*:

(1) $\psi|_{S_0} \subseteq \mathbb{R}^d \oplus 0 \oplus \mathbb{R}$ and induces an l.s.o.p. Θ_S of $\mathbb{R}[S]$ and an HL element ω_S of $\mathbb{R}[S]/(\Theta_S)$; $\psi|_{\sigma_0} \subseteq 0 \oplus \mathbb{R}^{d-2i_0-1} \oplus \mathbb{R}$ and induces an l.s.o.p. $\Theta_{\partial\sigma}$ of $\mathbb{R}[\partial\sigma]$ and an HL element $\omega_{\partial\sigma}$ of $\mathbb{R}[S]/(\Theta_{\partial\sigma})$. By Theorem 4.2.4, $\omega_S + \omega_{\partial\sigma}$ is an HL element of $\mathbb{R}[S * \partial\sigma]/(\Theta_S \cup \Theta_{\partial\sigma})$.

In matrix language, the first $2d - 2i_0 - 1$ columns of $[\psi|_{S_0 \cup \sigma_0}]$ form an l.s.o.p. of $\mathbb{R}[S * \partial\sigma]$, and its last column is the corresponding HL element.

(2) $0 \neq \psi(v_F) \in \mathbb{R}^d \oplus 0 \oplus \mathbb{R}$ induces a map $\pi : \mathbb{R}^{2d-2i_0} \to \mathbb{R}^{2d-2i_0}/\operatorname{span}\psi(v_F) \cong \mathbb{R}^{2d-2i_0-1}$ such that $\pi \circ \psi|_{S_0 \cup \sigma_0}$ induces an element in $\Omega(G, d - i_0 - 2)$ for $G = \{u\} * \mathrm{lk}(\{v_F, u\}, L)$.

To see this, consider e.g. an embedding ψ' with $\psi'(v_F) = (1, 0, ..., 0)$, $\psi'(u) = (0, 1, 0, ..., 0)$, $\psi'(s)$ vanishes on the first two coordinates for any $s \in S_0 \setminus \{u\}$ and in addition $[\psi']$ vanishes on all entries on which we required in (1) that $[\psi]$ vanishes. By Theorem 4.2.4 there exists such ψ' so that its composition with the projection $\pi' : \mathbb{R}^{2d-2i_0} \to \mathbb{R}^{2d-2i_0}/\operatorname{span}\{\psi(v_F), \psi(u)\}$ induces a pair (Θ, ω) of an l.s.o.p. and an HL element for $\mathrm{lk}(\{v_F, u\}, L) = \mathrm{lk}(F, S) * \partial(F \setminus \{u\}) * \partial\sigma$. By adding x_u to this l.s.o.p. we obtain an l.s.o.p.

for G where $\omega : H(G)_{d-i_0-2} \rightarrow H(G)_{d-i_0-1}$ is injective. Now perturb ψ' to obtain ψ for which property (2) hold.

The restriction of maps ψ with property (2) to $\mathrm{st}(F,S)_0 \cup \{v_F\}$ is a nonempty Zariski open set in the space of embeddings $\{f : \mathrm{st}(F,S)_0 \cup \{v_F\} \rightarrow \mathbb{R}^d \oplus 0 \oplus \mathbb{R}\}$. The restriction of maps ψ with property (1) to S_0 is a nonempty Zariski open set in the space of embeddings $\{f : S_0 \rightarrow \mathbb{R}^d \oplus 0 \oplus \mathbb{R}\}$. Hence, their projections on the linear subspace $\{f : \mathrm{st}(F,S)_0 \rightarrow \mathbb{R}^d \oplus 0 \oplus \mathbb{R}\}$ are nonempty Zariski open sets (in this subspace). The intersection of these projections is again a nonempty Zariski open set, thus there are maps ψ for which both properties (1) and (2) hold.

(3) $\psi|_{S_0 \cup \{v_F\}} \subseteq \mathbb{R}^d \oplus 0 \oplus \mathbb{R}$ and the first d columns of $[\psi]$ induce an l.s.o.p. Θ_T of $\mathbb{R}[T]$.

The set of restrictions $\psi|_{T_0}$ of maps ψ with property (3) is nonempty Zariski open in the subspace $\{f : T_0 \rightarrow \mathbb{R}^d \oplus 0 \oplus 0\}$; hence, so is its projection on the linear subspace $\{f : \mathrm{st}(F,S)_0 \rightarrow \mathbb{R}^d \oplus 0 \oplus 0\}$. By similar considerations to the above, there are maps ψ for which all the properties (1), (2) and (3) hold.

The proof of Proposition 4.6.1 together with properties (1) and (2) tell us that for small enough ϵ, the map $\psi" : L_0 \longrightarrow \mathbb{R}^{2d-2i_0}$ defined by $\psi"(v_F) = \psi(u) + \epsilon(\psi(v_F) - \psi(u))$ and $\psi"(v) = \psi(v)$ for every other vertex $v \in L_0$, satisfy $\mathrm{Ker}\, \psi"^{2d-2i_0}_L = 0$ (see equation (4.12) for the definition of this map). As a nonempty Zariski open set is dense, by looking on the subspace $\{f : T_0 \rightarrow \mathbb{R}^d \oplus 0 \oplus \mathbb{R}\}\}$, we can take $\psi(v_F)$ and ϵ such that $\psi"$ satisfies property (3) as well.

Thus, the first d columns of $[\psi"]$ induce an l.s.o.p. Θ_T of T, the next $d - i_0 - 1$ columns induce an l.s.o.p. $\Theta_{\partial\sigma}$ of $\partial\sigma$, and the last column of $[\psi"]$ is a $(d - i_0 - 1)$-Lefschetz element of $\mathbb{R}[L]/(\Theta_T \cup \Theta_{\partial\sigma})$. This contradicts our earlier conclusion, which was based on assuming that the assertion of this theorem is incorrect. $\square$

Corollary 4.6.6 *Let $\mathcal{S}$ be a family of homology spheres which is closed under taking links and such that all of its elements are HL. Let $\mathbb{S} = \mathbb{S}(\mathcal{S})$ be the family obtained from $\mathcal{S} \cup \{\partial\sigma^n : n \geq 1\}$ by taking the closure under the operations: (0) taking links; (1) join; (2) Stellar subdivisions. Then every element in $\mathbb{S}$ is HL.*

Proof : We prove by double induction - on dimension, and on the sequence of operations of type (0),(1) and (2) which define $S \in \mathbb{S}$ - that S and all its face links are HL. Let us call S with this property *hereditary HL*.

Note that every $S \in \mathcal{S}$, every boundary of a simplex, and every (homology) sphere of dimension ≤ 2, is hereditary HL. This provides the base of the induction.

Clearly if S is hereditary HL, then so are all of its links, as $\mathrm{lk}(Q, (\mathrm{lk}(F, S)) = \mathrm{lk}(Q \uplus F, S)$. If S and S' are hereditary HL then by Theorem 4.2.4 so is $S * S'$ (here we note that every $T \in S * S'$ is of the form $T = F \uplus F'$ where $F \in S$ and $F' \in S'$, and that $\mathrm{lk}(T, S * S') = \mathrm{lk}(F, S) * \mathrm{lk}(F', S')$). We are left to show that if $F \in S$ and S is hereditary HL, then so is $T := \mathrm{Stellar}(F, S)$. Assume $\dim F \geq 1$, otherwise there is nothing to prove. First we note that by the induction hypothesis for every $v \in T_0$, $\mathrm{lk}(v, T)$ is hereditary HL:
Case $v = v_F$: $\mathrm{lk}(v_F, T) = \mathrm{lk}(F, S) * \partial F$ is hereditary HL by Theorem 4.2.4, as argued above.
Case $v \in F$: $\mathrm{lk}(v, T) = \mathrm{Stellar}(F \setminus \{v\}, \mathrm{lk}(v, S))$ is hereditary HL by the induction hypothesis on the dimension.
Case $v \notin F$, $v \neq v_F$ and $F \in \mathrm{lk}(v, S)$: $\mathrm{lk}(v, T) = \mathrm{Stellar}(F, \mathrm{lk}(v, S))$ is hereditary HL by the induction hypothesis on the dimension.
Otherwise: $\mathrm{lk}(v, T) = \mathrm{lk}(v, S)$ is hereditary HL.

We are left to show that T is HL: S is HL, and for $u \in F$ $\mathrm{lk}(\{v_F, u\}, T) = \mathrm{lk}(F, S) * \partial(F \setminus \{u\})$ is HL by Theorem 4.2.4. Thus, by Theorem 4.6.5 T is HL, and together with the above, T is hereditary HL. $\square$

Remark: The barycentric subdivision of a simplicial complex K can be obtained by a sequence of Stellar subdivisions: order the faces of K of dimension > 0 by weakly decreasing size, and perform Stellar subdivisions at those faces according to this order; the barycentric subdivision of K is obtained. Brenti and Welker [11], Corollary 3.5, showed that the h-polynomial of the barycentric subdivision of a Cohen-Macaulay complex has only simple and real roots, and hence is unimodal. In particular, barycentric subdivision preserves non-negativity of the g-vector for spheres with all links being HL. The above corollary shows that the hereditary HL property itself is preserved.

4.7 Open problems

1. Show that for any $d \geq 1$ and any homology $2d$-sphere L, $\Omega(L, d)$ intersects every hyperplane in the vector space $\mathbb{R}^{|L_0| \times (2d+2)}$ of $2d + 2$ degree one forms.

   In Theorem 4.5.2 we have seen that to conclude the g-conjecture $\Omega(L, d) \neq \emptyset$ is enough, but this stronger conjecture may be easier to prove in the PL case, by using bistellar moves. It tries to correct an unpublished argument of Swartz.

2. *Shifting theoretic lower bound relation:* In Example 2.1.8 we computed the algebraic shifting of a stacked $(d-1)$-sphere on n vertices, denoted

$\Delta(S(d,n))$. Prove that if K is a homology $(d-1)$-sphere on n vertices then $\Delta(S(d,n)) \subseteq \Delta(K)$.

This conjecture immediately implies Barnette's lower bound theorem for triangulated spheres. The symmetric case of this conjecture is equivalent to the claim that the multiplication map $y_{d+1}^{d-2} : H(K)_1 \longrightarrow H(K)_{d-1}$, is an isomorphism. Rigidity theory only tells us that $y_{d+1} : H(K)_1 \longrightarrow H(K)_2$ is injective.

3. Is the join of a unimodal WL complex with an HL complex always unimodal WL?

4. Is ω from Theorem 4.4.1 an HL element for $K \# L$?

5. Let $\mathbb{S}$ be a family of simplicial complexes which is closed under links and joins and contains all boundaries of simplices (e.g. simplicial spheres, homology spheres, PL-spheres). Prove that if all elements of $\mathbb{S}$ are unimodal WL then all of them are HL.

Chapter 5

Algebraic Shifting and the g-Conjecture: a Topological Approach

5.1 Kalai-Sarkaria conjecture

As we have seen in Section 4.1, if a simplicial $(d-1)$-sphere K satisfies $\Delta(K) \subseteq \Delta(d)$ then $g(K)$ is an M-sequence. A stronger conjecture was stated, independently, by Kalai and Sarkaria [34], Conjecture 27:

Conjecture 5.1.1 *(Kalai, Sarkaria) If K is a simplicial complex with n vertices and $||K||$ can be embedded is the $(d-1)$-sphere, then $\Delta(K) \subseteq \Delta(d,n)$. Equivalently, $T_{d-k} \notin \Delta(K)$ for every $0 \leq k \leq \lfloor d/2 \rfloor$ (see equation (4.1)).*

Note that by Swartz lifting theorem, Theorem 4.5.2, to conclude Conjecture 4.1.1 for simplicial spheres it is enough to show that for d odd $T_{\lceil d/2 \rceil} := \{\lceil d/2 \rceil + 2, ..., d+2\} \notin \Delta(K)$. The later conjecture trivially holds for $d = 1$, as 3 points cannot be embedded into 2 points. It holds for $d = 3$ by Theorem 3.6.1, as $\{4,5\} \in \Delta(K)$ implies that the graph of K has a K_5-minor, hence K does not embed in S^2. It is open for $d = 5, 7, 9, ...$

Sarkaria suggested to relate the Van-Kampen obstruction to embeddability of $||K||$ to that of $||\Delta(K)||$. We recall this obstruction in the next section, and later relate it to a notion of minors for simplicial complexes, and to a combinatorial problem which would imply Conjecture 5.1.1.

5.2 Van Kampen's Obstruction

5.2.1 Deleted join and $\mathbb{Z}_2$ coefficients

The presentation here is based on work of Sarkaria [61, 60] who attributes it to Wu [78] and all the way back to Van Kampen [35]. It is a Smith theoretic interpretation of Van Kampen's obstructions.

Let K be a simplicial complex. The join $K * K$ is the simplicial complex $\{S^1 \uplus T^2 : S, T \in K\}$ (the superscript indicates two disjoint copies of K). The *deleted join* K_* is the subcomplex $\{S^1 \uplus T^2 : S, T \in K, S \cap T = \emptyset\}$. The restriction of the involution $\tau : K * K \longrightarrow K * K$, $\tau(S^1 \cup T^2) = T^1 \cup S^2$ to K_* is into K_*. It induces a $\mathbb{Z}_2$-action on the cochain complex $C^*(K_*; \mathbb{Z}_2)$. For a simplicial cochain complex C over $\mathbb{Z}_2$ with a $\mathbb{Z}_2$-action τ, let C_S be its subcomplex of *symmetric cochains*, $\{c \in C : \tau(c) = c\}$. Restriction induces an action of τ as the identity map on C_S. Note that the following sequence is exact in dimensions ≥ 0:

$$0 \longrightarrow C_S(K_*) \longrightarrow C(K_*) \xrightarrow{id+\tau} C_S(K_*) \longrightarrow 0$$

where $C_S(K_*) \longrightarrow C(K_*)$ is the trivial injection. (The only part of this statement that may be untrue for a non-free simplicial cochain complex C over $\mathbb{Z}_2$ with a $\mathbb{Z}_2$-action τ, is that $id + \tau$ is surjective.) Thus, there is an induced long exact sequence in cohomology

$$H^0_S(K_*) \xrightarrow{\mathrm{Sm}} H^1_S(K_*) \longrightarrow ... \longrightarrow H^q_S(K_*) \longrightarrow H^q(K_*) \longrightarrow H^q_S(K_*) \xrightarrow{\mathrm{Sm}} H^{q+1}_S(K_*) \longrightarrow ....$$

Composing the connecting homomorphism Sm m times we obtain a map $\mathrm{Sm^m : H^0_S(K_*) \longrightarrow H^m_S(K_*)}$. For the fundamental 0-cocycle 1_{K_*}, i.e. the one which maps $\sum_{v \in (K_*)_0} a_v v \mapsto \sum_{v \in (K_*)_0} a_v \in \mathbb{Z}_2$, let $[1_{K_*}]$ denotes its image in $H^0_S(K_*)$. $\mathrm{Sm^m([1_{K_*}])}$ is called the m-th *Smith characteristic class* of K_*, denoted also as $\mathrm{Sm^m(K)}$.

For any positive integer d let $H(d)$ be the $(d-1)$-skeleton of the $2d$-dimensional simplex. A well known result by Van Kampen and Flores [23, 35] asserts that the Van Kampen obstruction with $\mathbb{Z}$ coefficients (see the next subsection) of $H(d)$ in dimension $(2d-1)$ does not vanish, and hence $H(d)$ is not embeddable in the $2(d-1)$-sphere (note that the case $H(2) = K_5$ is part of the easier direction of Kuratowski's theorem). Here are the analogous statements for $\mathbb{Z}_2$ coefficients.

Theorem 5.2.1 *(Sarkaria [60] Theorem 6.5, see also Wu [78] pp.114-118.) For every $d \geq 1$, $\mathrm{Sm^{2d-1}(1_{H(d)_*})} \neq 0$.*

Theorem 5.2.2 *(Sarkaria [60] Theorem 6.4 and [61] p.6) If a simplicial complex K embeds in $\mathbb{R}^m$ (or in the m-sphere) then* $\mathrm{Sm^{m+1}(1_{K_*})} = 0$.

Sketch of proof: The definition of Smith class makes sense for singular homology as well; the obvious map from the simplicial chain complex to the singular one induces an isomorphism between the corresponding Smith classes. The definition of deleted join makes sense for subspaces of a Euclidean space as well (see e.g. [44], 5.5); thus an embedding $||K||$ of K into $\mathbb{R}^m$ induces a continuous $\mathbb{Z}_2$-map from $||K||_*$ into the join of $\mathbb{R}^m$ with itself minus the diagonal, which is $\mathbb{Z}_2$-homotopic to the antipodal m-sphere, S^m. The equivariant cohomology of S^m over $\mathbb{Z}_2$ is isomorphic to the ordinary cohomology of $\mathbb{R}P^m$ over $\mathbb{Z}_2$, which vanishes in dimension $m+1$. We get that $\mathrm{Sm^{m+1}(S^m)}$ maps to $\mathrm{Sm^{m+1}(1_{||K||_*})}$ and hence the later equals to zero as well. But $||K_*||$ and $||K||_*$ are $\mathbb{Z}_2$-homotopic, hence $\mathrm{Sm^{m+1}(1_{K_*})} = 0$. $\square$

5.2.2 Deleted product and $\mathbb{Z}$ coefficients

More commonly in the literature, Van Kampen's obstruction is defined via deleted products and with $\mathbb{Z}$ coefficients, where, except for 2-simplicial complexes, its vanishing is also sufficient for embedding of the complex in a Euclidean space of double its dimension.

The presentation of the background on the obstruction here is based on the ones in [56], [78] and [74].

Let K be a finite simplicial complex. Its deleted product is $K \times K \setminus \{(x,x) : x \in K\}$, employed with a fixed-point free $\mathbb{Z}_2$-action $\tau(x,y) = (y,x)$. It $\mathbb{Z}_2$-deformation retracts into $K_\times = \cup\{S \times T : S, T \in K, S \cap T = \emptyset\}$, with which we associate a cell chain complex over $\mathbb{Z}$: $C_\bullet(K_\times) = \bigoplus\{\mathbb{Z}(S \times T) : S \times T \in K_\times\}$ with a boundary map $\partial(S \times T) = \partial S \times T + (-1)^{\mathrm{dim S}} S \times \partial T$, where $S \times T$ is a $\mathrm{dim(S \times T)}$-chain. The dual cochain complex consists of the j-cochains $C^j(K_\times) = \mathrm{Hom_{\mathbb{Z}}(C_j(K_\times), \mathbb{Z})}$ for every j.

There is a $\mathbb{Z}_2$-action on $C_\bullet(K_\times)$ defined by $\tau(S \times T) = (-1)^{\mathrm{dim(S)dim(T)}} T \times S$. As it commutes with the coboundary map, by restriction of the coboundary map we obtain the subcomplexes of symmetric cochains $C_s^\bullet(K_\times) = \{c \in C^\bullet(K_\times) : \tau(c) = c\}$ and of antisymmetric cochains $C_a^\bullet(K_\times) = \{c \in C^\bullet(K_\times) : \tau(c) = -c\}$. Their cohomology rings are denoted by $H_s^\bullet(K_\times)$ and $H_a^\bullet(K_\times)$ respectively. Let H_{eq}^m be H_s^m for m even and H_a^m for m odd.

For every finite simplicial complex K there is a unique $\mathbb{Z}_2$-map, up to $\mathbb{Z}_2$-homotopy, into the infinite dimensional sphere $i : K_\times \to S^\infty$, and hence a uniquely defined map $i^* : H_{\mathrm{eq}}^\bullet(S^\infty) \to H_{\mathrm{eq}}^\bullet(K_\times)$. For z a generator of $H_{\mathrm{eq}}^m(S^\infty)$ call $o^m = o_{\mathbb{Z}}^m(K_\times) = i^*(z)$ the Van Kampen obstruction; it is uniquely defined up to a sign. It turns out to have the following explicit description: fix a

total order $<$ on the vertices of K. It evaluates elementary symmetric chains of even dimension $2m$ by

$$o^{2m}((1+\tau)(S\times T)) = \begin{cases} 1 \text{ if the unordered pair } \{S,T\} \text{ is of the form } s_0<t_0<..<s_m<t_m \\ 0 \text{ for other pairs } \{S,T\} \end{cases} \tag{5.1}$$

and evaluates elementary antisymmetric chains of odd dimension $2m+1$ by

$$o^{2m+1}((1-\tau)(S\times T)) = \begin{cases} 1 \text{ if } \{S,T\} \text{ is of the form } t_0<s_0<t_1<..<t_m<s_m<t_{m+1} \\ 0 \text{ for other pairs } \{S,T\} \end{cases} \tag{5.2}$$

where the s_l's are elements of S and the t_l's are elements of T. Its importance to embeddability is given in the following classical result:

Theorem 5.2.3 *[35, 64, 78] If a simplicial complex K embeds in $\mathbb{R}^m$ then $H^{\bullet}_{\text{eq}}(K_{\times}) \ni o^m_{\mathbb{Z}}(K_{\times}) = 0$. If K is m-dimensional and $m \neq 2$ then $o^{2m}_{\mathbb{Z}}(K_{\times}) = 0$ implies that K embeds in $\mathbb{R}^{2m}$.*

5.3 Relation to minors of simplicial complexes

5.3.1 Definition of minors and statement of results

The concept of graph minors has proved be to very fruitful. A famous result by Kuratowski asserts that a graph can be embedded into a 2-sphere if and only if it contains neither of the graphs K_5 and $K_{3,3}$ as minors. We wish to generalize the notion of graph minors to all (finite) simplicial complexes in a way that would produce analogous statements for embeddability of higher dimensional complexes in higher dimensional spheres. We hope that these higher minors will be of interest in future research, and indicate some results and problems to support this hope.
Let K and K' be simplicial complexes. $K \mapsto K'$ is called a *deletion* if K' is a subcomplex of K. $K \mapsto K'$ is called an *admissible contraction* if K' is obtained from K by identifying two distinct vertices of K, v and u, such that v and u are not contained in any missing face of K of dimension $\leq$ dim(K). (A set T is called a missing face of K if it is not an element of K while all its proper subsets are.) Specifically, $K' = \{T : u \notin T \in K\} \cup \{(T \setminus \{u\}) \cup \{v\} : u \in T \in K\}$. An equivalent formulation of the condition for admissible contractions is that the following holds:

$$(\mathrm{lk(v,K)} \cap \mathrm{lk(u,K)})_{\mathrm{dim(K)}-2} = \mathrm{lk(\{v,u\},K)}. \tag{5.3}$$

For K a graph, (5.3) just means that $\{v, u\}$ is an edge in K.

We say that a simplicial complex H is a *minor* of K, and denote it by $H < K$, if H can be obtained from K by a sequence of admissible contractions and deletions (the relation $<$ is a partial order). Note that for graphs

this is the usual notion of a minor.

Remarks: (1) In equation (5.3), the restriction to the skeleton of dimension at most $\mathrm{dim(K)} - 2$ can be relaxed by restriction to the skeleton of dimension at most $\min\{\mathrm{dim(lk(u, K))}, \mathrm{dim(lk(v, K))}\} - 1$, making the condition for admissible contraction *local*, and weaker. All the results and proofs in this section hold verbatim for this notion of a minor as well.

(2) In the definition of a minor, without loss of generality we may replace the local condition from the remark above by the following stronger local condition, called the *Link Condition* for $\{u, v\}$:

$$\mathrm{lk(u, K)} \cap \mathrm{lk(v, K)} = \mathrm{lk(\{u, v\}, K)}. \tag{5.4}$$

To see this, let $K \mapsto K'$ be an admissible contraction which is obtained by identifying the vertices u and v where $\mathrm{dim(lk(u, K))} \leq \mathrm{dim(lk(v, K))}$. Delete from K all the faces $F \uplus \{u\}$ such that $F \uplus \{u, v\}$ is a missing face of dimension $\mathrm{dim(lk(u, K))} + 2$, to obtain a simplicial complex L. Note that $\{u, v\}$ satisfies the Link Condition in L, and the identification of u with v in L results in K'. I thank an anonymous referee for this remark.

We first relate this minor notion to Van Kampen's obstruction with $\mathbb{Z}_2$ coefficients.

Theorem 5.3.1 *Let H and K be simplicial complexes. If $H < K$ and* $\mathrm{Sm^m(H)} \neq 0$ *then* $\mathrm{Sm^m(K)} \neq 0$.

Corollary 5.3.2 *For every $d \geq 1$, if $H(d) < K$ then K is not embeddable in the $2(d-1)$-sphere.* $\square$

Remark: Corollary 5.3.2 would also follow from the following conjecture:

Conjecture 5.3.3 *If $H < K$ and K is embeddable in the m-sphere then H is embeddable in the m-sphere.*

The analogue of Theorem 5.3.1 with $\mathbb{Z}$ coefficients holds:

Theorem 5.3.4 *Let H and K be simplicial complexes. If $H < K$ and $o^m_{\mathbb{Z}}(H_\times) \neq 0$ then $o^m_{\mathbb{Z}}(K_\times) \neq 0$.*

From Theorems 5.3.4 and 5.2.3 it follows that Conjecture 5.3.3 is true when $\mathrm{2dim(H) = m} \neq 4$ (and, trivially, when $\mathrm{2dim(H) < m}$).

In view of Theorems 5.3.1, 5.2.1 and 5.2.2, to conclude Conjecture 5.1.1 in the symmetric case $T_{\lceil d/2 \rceil}$ where d odd, and hence also the g-conjecture for simplicial spheres Conjecture 4.1.1, it suffices to prove the following combinatorial conjecture:

Conjecture 5.3.5 *Let K be a simplicial complex. For every $d \geq 1$, if $H(d) \subseteq \Delta(K)$ then $H(d) < K$.*

This conjecture holds for $d = 1, 2$ and is otherwise open. Assume that d_0 is the minimal d for which Conjecture 5.3.5 fails, and let K be a minimal counterexample w.r.t. the number of vertices. W.l.o.g. $\mathrm{dim(K) = d_0 - 1}$. Then $H(d_0) \subseteq \Delta(K)$ but $H(d_0) \not< K$. We may assume further that

(1) (Maximality) For every missing face T of K of dimension $\leq d_0 - 1$, $H(d_0) < K \cup \{T\}$.

(2) (Links) For every two distinct vertices $v, u \in K_0$ such that $\{v, u\}$ is not contained in any missing face of K of dimension $\leq d_0 - 1$, $H(d_0 - 1) < \mathrm{lk_K(u)} \cap \mathrm{lk_K(v)}$.

(1) follows from the fact that if $K \subseteq L$ then $\Delta(K) \subseteq \Delta(L)$. (2) follows from the minimality and from Proposition 4.6.1. Indeed, if $H(d_0 - 1) \not< \mathrm{lk_K(u)} \cap \mathrm{lk_K(v)}$ then $H(d_0 - 1) \nsubseteq \Delta(\mathrm{lk_K(u)} \cap \mathrm{lk_K(v)})$ and as $H(d_0) \subseteq \Delta(K)$ we obtain that the contraction of v, u results in K' for which $H(d_0) \subseteq \Delta(K')$ and $H(d_0) \not< K'$, contradicting the minimality of K.

This led us to suspect that counterexamples for $d = 3$ may be provided by the following complexes. Let M_L be the vertex transitive neighborly 4-sphere on 15 vertices manifold_$(4, 15, 5, 1)$ found by Frank Lutz [41]. Note that every edge in M_L is contained in a missing triangle. Let K be the 2-skeleton of M_L union with a missing triangle. It is easy to find triangles such that every edge in K is contained in a missing triangle. As M_L is neighborly, by counting and the fact that $\Delta(K)$ is shifted, we conclude that $\{5, 6, 7\} \in \Delta(K)$, hence $H(3) \subseteq \Delta(K)$. Is $H(d) < K$? Note that deletions must be performed before any contraction is possible.

5.3.2 Proof of Theorem 5.3.1

The idea is to define an injective chain map $\phi : C_*(H; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$ which induces $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{H_*})}$ for every $m \geq 0$.

Lemma 5.3.6 *Let $K \mapsto K'$ be an admissible contraction. Then it induces an injective chain map $\phi : C_*(K'; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$.*

Proof: Fix a labeling of the vertices of K, $v_0, v_1, .., v_n$, such that K' is obtained from K by identifying $v_0 \mapsto v_1$ where $\mathrm{dim(lk(v_0, K)) \leq dim(lk(v_1, K))}$.

Let $F \in K'$. If $F \in K$, define $\phi(F) = F$. If $F \notin K$, define $\phi(F) = \sum\{(F \setminus v) \cup v_0 : v \in F, (F \setminus v) \cup v_0 \in K\}$. Note that if $F \notin K$ then $v_1 \in F$ and $(F \setminus v_1) \cup v_0 \in K$, so the sum above is nonzero. Extend linearly to obtain a map $\phi : C_*(K'; \mathbb{Z}_2) \longrightarrow C_*(K; \mathbb{Z}_2)$.

First, let us check that ϕ is a chain map, i.e. that it commutes with the boundary maps ∂. It is enough to verify this for the basis elements F where $F \in K'$. If $F \in K$ then $\mathrm{supp}(\partial F) \subseteq K$, hence $\partial(\phi F) = \partial F = \phi(\partial F)$. If $F \notin K$ then $\partial(\phi F) = \partial(\sum\{(F \setminus v) \cup v_0 : v \in F, (F \setminus v) \cup v_0 \in K\})$, and as we work over $\mathbb{Z}_2$, this equals

$$\partial(\phi F) = \sum\{F \setminus v : v \in F, (F \setminus v) \cup v_0 \in K\} + \tag{5.5}$$

$$\sum\{(F \setminus \{u, v\}) \cup v_0 : u, v \in F, (F \setminus v) \cup v_0 \in K, (F \setminus u) \cup v_0 \notin K\}.$$

On the other hand $\phi(\partial F) = \phi(\sum\{F \setminus u : u \in F, F \setminus u \in K\}) + \phi(\sum\{F \setminus u : u \in F, F \setminus u \notin K\})$ and as we work over $\mathbb{Z}_2$, this equals

$$\phi(\partial F) = \sum\{F \setminus u : u \in F, (F \setminus u) \in K\} + \tag{5.6}$$

$$\sum\{(F \setminus \{u, v\}) \cup v_0 : u, v \in F, (F \setminus \{u, v\}) \cup v_0 \in K, (F \setminus v) \in K, (F \setminus u) \notin K\}.$$

It suffices to show that in equations (5.5) and (5.6) the left summands on the RHSs are equal, as well as the right summands on the RHSs. This follows from observation 5.3.7 below. Thus ϕ is a chain map.

Second, let us check that ϕ is injective. Let π_K be the restriction map $C_*(K'; \mathbb{Z}_2) \longrightarrow \oplus\{\mathbb{Z}_2 F : F \in K' \cap K\}$, $\pi_K(\sum\{\alpha_F F : F \in K'\}) = \sum\{\alpha_F F : F \in K' \cap K\}$. Similarly, let $\pi_K^{\perp}$ be the restriction map $C_*(K'; \mathbb{Z}_2) \longrightarrow \oplus\{\mathbb{Z}_2 F : F \in K' \backslash K\}$. Note that for a chain $c \in C_*(K'; \mathbb{Z}_2)$, $c = \pi_K(c) + \pi_K^{\perp}(c)$ and $\mathrm{supp}(\phi(\pi_K(c))) \cap \mathrm{supp}(\phi(\pi_K^{\perp}(c))) = \emptyset$. Assume that $c_1, c_2 \in C_*(K'; \mathbb{Z}_2)$ such that $\phi(c_1) = \phi(c_2)$. Then $\pi_K(c_1) = \phi(\pi_K(c_1)) = \phi(\pi_K(c_2)) = \pi_K(c_2)$, and $\phi(\pi_K^{\perp}(c_1)) = \phi(\pi_K^{\perp}(c_2))$. Note that if $F_1, F_2 \notin K$ then $F_1, F_2 \in K'$ and if $F_1 \neq F_2$ then $\mathrm{supp}(\phi(1F_1)) \ni (F_1 \setminus v_1) \cup v_0 \notin \mathrm{supp}(\phi(1F_2))$. Hence also $\pi_K^{\perp}(c_1) = \pi_K^{\perp}(c_2)$. Thus $c_1 = c_2$. $\square$

Observation 5.3.7 *Let $K \mapsto K', v_0 \mapsto v_1$ be an admissible contraction with* $\mathrm{dim}(\mathrm{lk}(\mathrm{v_0}, \mathrm{K})) \leq \mathrm{dim}(\mathrm{lk}(\mathrm{v_1}, \mathrm{K}))$. *Let $K' \ni F \notin K$ and $v \in F$. Then $(F \backslash v) \in K$ if and only if $(F \setminus v) \cup v_0 \in K$.*

Proof: Assume $F \setminus v \in K$. As $(F \setminus v_1) \cup v_0 \in K$ we only need to check the case $v \neq v_1$. We proceed by induction on dim(F). As $\{v_0, v_1\} \in K$ whenever $\mathrm{dim(K)} > 0$ (and whenever $\mathrm{dim}(\mathrm{lk}(\mathrm{v_0}, \mathrm{K})) \geq 0$, if we use the weaker local condition for admissible contractions), the case $\mathrm{dim(F)} \leq 1$ is clear. (If $\mathrm{dim(K)} = 0$ there is nothing to prove. For the weaker local condition for admissible contractions, if $\mathrm{lk}(\mathrm{v_0}, \mathrm{K})) = \emptyset$ then there is nothing to prove.) By the induction hypothesis we may assume that all the proper subsets of $(F \setminus v) \cup v_0$ are in K. Also $v_0, v_1 \in (F \setminus v) \cup v_0$. The admissibility of the contraction implies that $(F \setminus v) \cup v_0 \in K$. The other direction is trivial. $\square$

Lemma 5.3.8 *Let $\phi : C_*(K';\mathbb{Z}_2) \longrightarrow C_*(K;\mathbb{Z}_2)$ be the injective chain map defined in the proof of Lemma 5.3.6 for an admissible contraction $K \mapsto K'$. Then for every $m \geq 0$, $\phi^*(\mathrm{Sm^m([1_{K_*}])}) = \mathrm{Sm^m([1_{K'_*}])}$ for the induced map ϕ^*.*

Proof: For two simplicial complexes L and L' and a field k, the following map is an isomorphism of chain complexes:

$$\alpha = \alpha_{L,L',k} : C(L;k)\otimes_k C(L';k) \longrightarrow C(L*L';k),\ \ \alpha((1T)\otimes(1T')) = 1(T\uplus T')$$

where $T \in L, T' \in L'$ and α is extended linearly. In case $L = L'$ (in the definition of join we think of L and L' as two disjoint copies of L) and k is understood we denote $\alpha_{L,L',k} = \alpha_L$.

Thus there is an induced chain map $\phi_* : C_*(K'*K';\mathbb{Z}_2) \longrightarrow C_*(K*K;\mathbb{Z}_2)$, $\phi_* = \alpha_K \circ \phi\otimes\phi\circ\alpha_{K'}^{-1}$ where $\phi\otimes\phi : C(K';\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} C(K';\mathbb{Z}_2) \longrightarrow C(K;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} C(K;\mathbb{Z}_2)$ is defined by $\phi\otimes\phi(c\otimes c') = \phi(c)\otimes\phi(c')$ (which this is a chain map).

Consider the subcomplex $C_*(K'_*;\mathbb{Z}_2) \subseteq C_*(K' * K';\mathbb{Z}_2)$. We now verify that every $c \in C_*(K'_*;\mathbb{Z}_2)$ satisfies $\phi_*(c) \in C_*(K_*;\mathbb{Z}_2)$. It is enough to check this for chains of the form $c = 1(S^1 \cup T^2)$ where $S, T \in K'$ and $S \cap T = \emptyset$. For a collection of sets A let $V(A) = \cup_{a\in A} a$. Clearly if the condition

$$V(\mathrm{supp}(\phi(S))) \cap V(\mathrm{supp}(\phi(T))) = \emptyset \tag{5.7}$$

is satisfied then we are done. If $v_1 \notin S, v_1 \notin T$, then $\phi(S) = S, \phi(T) = T$ and (5.7) holds. If $T \ni v_1 \notin S$, then $\phi(S) = S$ and $V(\mathrm{supp}\,\phi(T)) \subseteq T \cup \{v_0\}$. As $v_0 \notin S$ condition (5.7) holds. By symmetry, (5.7) holds when $S \ni v_1 \notin T$ as well.

With abuse of notation (which we will repeat) we denote the above chain map by ϕ, $\phi : C_*(K'_*;\mathbb{Z}_2) \longrightarrow C_*(K_*;\mathbb{Z}_2)$. For a simplicial complex L, the involution $\tau_L : L_* \longrightarrow L_*$, $\tau_L(S^1 \cup T^2) = T^1 \cup S^2$ induces a $\mathbb{Z}_2$-action on $C_*(L_*;\mathbb{Z}_2)$. It is immediate to check that $\alpha_{L,L',k}$ and $\phi\otimes\phi$ commute with these $\mathbb{Z}_2$-actions, and hence so does their composition, ϕ. Thus, we have proved that $\phi : C_*(K'_*;\mathbb{Z}_2) \longrightarrow C_*(K_*;\mathbb{Z}_2)$ is a $\mathbb{Z}_2$-chain map.

Therefore, there is an induced map on the symmetric cohomology rings $\phi : H^*_S(K_*) \longrightarrow H^*_S(K'_*)$ which commutes with the connecting homomorphisms $\mathrm{Sm : H^i_S(L) \longrightarrow H^{i+1}_S(L)}$ for $L = K_*, K'_*$.

Let us check that for the fundamental 0-cocycles $\phi([1_{K_*}]) = [1_{K'_*}]$ holds. A representing cochain is $1_{K_*} : \oplus_{v\in(K_*)_0}\mathbb{Z}_2 v \longrightarrow \mathbb{Z}_2$, $1_{K_*}(1v) = 1$. As $\phi|_{C_0(K'_*)} = id$ (w.r.t. the obvious injection $(K'_*)_0 \longrightarrow (K_*)_0$), for every $u \in (K'_*)_0$ $(\phi 1_{K_*})(u) = 1_{K_*}(\phi|_{C_0(K'_*)}(u)) = 1_{K_*}(u) = 1$, thus $\phi(1_{K_*}) = 1_{K'_*}$.

As ϕ commutes with the Smith connecting homomorphisms, for every $m \geq 0$, $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{K'_*})}$. $\square$

Theorem 5.3.9 *Let H and K be simplicial complexes. If $H < K$ then there exists an injective chain map $\phi : C_*(H;\mathbb{Z}_2) \longrightarrow C_*(K;\mathbb{Z}_2)$ which induces $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{H_*})}$ for every $m \geq 0$.*

Proof: Let the sequence $K = K^0 \mapsto K^1 \mapsto ... \mapsto K^t = H$ demonstrate the fact that $H < K$. If $K^i \mapsto K^{i+1}$ is an admissible contraction, then by Lemmas 5.3.6 and 5.3.8 it induces an injective chain map $\phi_i : C_*(K^{i+1};\mathbb{Z}_2) \longrightarrow C_*(K^i;\mathbb{Z}_2)$ which in turn induces $\phi_i(\mathrm{Sm^m(1_{(K^i)_*})}) = \mathrm{Sm^m(1_{(K^{i+1})_*})}$ for every $m \geq 0$. If $K^i \mapsto K^{i+1}$ is a deletion - take ϕ_i to be the map induced by inclusion, to obtain the same conclusions. Thus, the composition $\phi = \phi_0 \circ ... \circ \phi_{t-1} : C_*(H;\mathbb{Z}_2) \longrightarrow C_*(K;\mathbb{Z}_2)$ is as desired. $\square$

Proof of Theorem 5.3.1: By Theorem 5.3.9 $\phi(\mathrm{Sm^m(1_{K_*})}) = \mathrm{Sm^m(1_{H_*})}$. Thus if $\mathrm{Sm^m(1_{H_*})} \neq 0$ then $\mathrm{Sm^m(1_{K_*})} \neq 0$. $\square$
Remark: The conclusion of Theorem 5.3.1 would fail if we allow arbitrary identifications of vertices. For example, let $K' = K_5$ and let K be obtained from K' by splitting a vertex $w \in K'$ into two new vertices u, v, and connecting u to a non-empty proper subset of $\mathrm{skel_0(K') \setminus \{w\}}$, denoted by A, and connecting v to $\mathrm{(skel_0(K') \setminus \{w\}) \setminus A}$. As K embeds into the 2-sphere, $\mathrm{Sm^3(K)} = 0$. By identifying u with v we obtain K', but $\mathrm{Sm^3(K')} \neq 0$. To obtain from this example an example where the edge $\{u, v\}$ is present, let $L = \mathrm{Cone}(K) \cup \{u, v\}$, and let L' be the complex obtained form L by identifying u with v. Then $\mathrm{Sm^4(L)} = 0$ while $\mathrm{Sm^4(L')} \neq 0$.

Example 5.3.10 *Let K be the simplicial complex spanned by the following collection of 2-simplices: $(\binom{[7]}{3} \setminus \{127, 137, 237\}) \cup \{128, 138, 238, 178, 278, 378\}$.*

K is not a subdivision of $H(3)$, and its geometric realization even does not contain a subspace homeomorphic to $H(3)$ (as there are no 7 points in $||K||$, each with a neighborhood whose boundary contains a subspace which is homeomorphic to K_6). Nevertheless, contraction of the edge 78 is admissible and results in $H(3)$. By Theorem 5.3.1 K has a non-vanishing Van Kampen's obstruction in dimension 5, and hence is not embeddable in the 4-sphere.

Example 5.3.11 *Let T be a missing d-face in the cyclic $(2d+1)$-polytope on n vertices, denoted by $C(2d+1, n)$, and let $K = (C(2d+1, n))_d \cup \{T\}$. Then $\Delta(K) \nsubseteq \Delta(2d+1, n)$ and K is not embeddable in the $2d$-sphere.*

Proof - sketch: As $\Delta(K)$ is shifted, by counting the number of faces not greater or equal $\{d+3, ..., 2d+3\}$ in the product partial order on $(d+1)$-tuples of $[n]$ we get $\Delta(K) \ni \{d+3, ..., 2d+3\} \notin \Delta(2d+1, n)$.

By Gale evenness condition (see e.g. [27],[81]), T is of the form $\{t_1, ..., t_{d+1}\}$ where $1 < t_1, t_{d+1} < n$ and $t_i + 1 < t_{i+1}$ for every i. Let $s_i = t_i + 1$ for $1 \leq i \leq d$, $s_0 = 1$ and $s_{d+1} = n$. In K we can admissibly contract $j+1 \mapsto j$ for $j, j+1 \in [s_{i-1}, t_i - 1]$ for $1 \leq i \leq d$ and similarly $j \mapsto j+1$ for $j, j+1 \in [t_{d+1}, s_{d+1}]$; to get a simplicial complex K' with the same description as K but on one vertex less, i.e. that its missing faces are the $(d+1)$-tuples of pairwise non-adjacent vertices bigger than the smallest vertex and smaller than the largest vertex except for one such set - T. Successive application of these contractions results in $H(d+1)$ as a minor of K, hence by Theorems 5.3.1, 5.2.1 and 5.2.2, K is not embeddable in the $2d$-sphere. $\square$

Example 5.3.11 is a special case of the following conjecture, a work in progress of Uli Wagner and the author.

Conjecture 5.3.12 *Let K be a triangulated $2d$-sphere and let T be a missing d-face in K. Let $L = K_d \cup \{T\}$. Then L does not embed in $\mathbb{R}^{2d}$.*

5.3.3 Proof of Theorem 5.3.4

Fix a total order on the vertices of K, $v_0 < v_1 < .. < v_n$ and consider an admissible contraction $K \mapsto K'$ where K' is obtained from K by identifying $v_0 \mapsto v_1$ (shortly this will be shown to be without loss of generality). Define a map ϕ as follows: for $F \in K'$

$$\phi(F) = \{^{F \quad \text{if } F\in K}_{\sum\{\mathrm{sgn}(v,F)(F\setminus v)\cup v_0:\ v\in F,(F\setminus v)\cup v_0\in K\} \quad \text{if } F\notin K} \tag{5.8}$$

where $\mathrm{sgn}(v,F) = (-1)^{|\{t\in F: t<v\}|}$. Extend linearly to obtain an injective $\mathbb{Z}$-chain map $\phi : C_\bullet(K') \longrightarrow C_\bullet(K)$. (The check that this map is indeed an injective $\mathbb{Z}$-chain map is similar to the proof of Lemma 5.3.6.) In case we contract a general $a \mapsto b$, for the signs to work out consider the map $\tilde{\phi} = \pi^{-1}\phi\pi$ rather than ϕ, where π is induced by a permutation on the vertices which maps $\pi(a) = v_0$, $\pi(b) = v_1$. Then $\tilde{\phi}$ is an injective $\mathbb{Z}$-chain map.

As $\phi(S \times T) := \phi(S) \times \phi(T)$ commutes with the $\mathbb{Z}_2$ action and with the boundary map on the chain complex of the deleted product, ϕ induces a map $H^\bullet_{\mathrm{eq}}(K_\times) \to H^\bullet_{\mathrm{eq}}(K'_\times)$. It satisfies $\phi^*(o^m_{\mathbb{Z}}(K_\times)) = o^m_{\mathbb{Z}}(K'_\times)$ for all $m \geq 1$. The checks are straightforward (for proving the last statement, choose a total order with contraction which identifies the minimal two elements $v_0 \mapsto v_1$, and show equality on the level of cochains). We omit the details.

If $K \mapsto K'$ is a deletion, consider the injection $\phi : K' \to K$ to obtain again an induced map with $\phi^*(o^m_{\mathbb{Z}}(K_\times)) = o^m_{\mathbb{Z}}(K'_\times)$.

Let the sequence $K = K^0 \mapsto K^1 \mapsto ... \mapsto K^t = H$ demonstrate the fact that $H < K$. By composing the corresponding maps as above we obtain a map ϕ^* with $\phi^*(o^m_{\mathbb{Z}}(K_\times)) = o^m_{\mathbb{Z}}(H_\times)$ and the result follows. $\square$

5.4 Topology preserving edge contractions

5.4.1 PL manifolds

The following theorem answers in the affirmative a question asked by Dey et. al. [19], who already proved the dimension ≤ 3 case.

Theorem 5.4.1 *Given an edge in a triangulation of a compact PL (piece-wise linear)-manifold without boundary, its contraction results in a PL-homeomorphic space if and only if it satisfies the Link Condition (5.4).*

Proof: Let M be a PL-triangulation of a compact d-manifold without boundary. Let ab be an edge of M and let M' be obtained from M by contracting $a \mapsto b$. We will prove that if the Link Condition (5.4) holds for ab then M and M' are PL-homeomorphic, and otherwise they are not homeomorphic (not even 'locally homologic'). For $d = 1$ the assertion is clear. Assume $d > 1$.

Denote $B(b) = \{b\} * \mathrm{ast}(b, \mathrm{lk}(a, M))$ and $L = \mathrm{ast}(a, M) \cap B(b)$. Then $M' = \mathrm{ast}(a, M) \cup_L B(b)$. As M is a PL-manifold without boundary, $\mathrm{lk}(a, M)$ is a $(d-1)$-PL-sphere (see e.g. [29], Corollary 1.16). By Newman's theorem (e.g. [29], Theorem 1.26) $\mathrm{ast}(b, \mathrm{lk}(a, M))$ is a $(d-1)$-PL-ball. Thus $B(b)$ is a d-PL-ball. Observe that $\partial(B(b)) = \mathrm{ast}(b, \mathrm{lk}(a, M)) \cup \{b\} * \mathrm{lk}(b, \mathrm{lk}(a, M)) = \mathrm{lk}(a, M) = \partial(\overline{\mathrm{st}}(a, M))$.

The identity map on $\mathrm{lk(a, M)}$ is a PL-homeomorphism $h : \partial(B(b)) \to \partial(\overline{\mathrm{st}}(a, M))$, hence it extends to a PL-homeomorphism $\tilde{h} : B(b) \to \overline{\mathrm{st}}(a, M)$ (see e.g. [29], Lemma 1.21).

Note that $L = \mathrm{lk(a, M)} \cup (\{\mathrm{b}\} * (\mathrm{lk(a, M)} \cap \mathrm{lk(b, M)}))$.

If $\mathrm{lk(a)} \cap \mathrm{lk(b)} = \mathrm{lk(ab)}$ (in M) then $L = \mathrm{lk(a, M)}$, hence gluing together the maps $\tilde{h}$ and the identity map on $\mathrm{ast(a, M)}$ results in a PL-homeomorphism from M' to M.

If $\mathrm{lk(a)} \cap \mathrm{lk(b)} \neq \mathrm{lk(ab)}$ (in M) then $\mathrm{lk(a, M)} \subsetneq \mathrm{L}$. The case $L = B(b)$ implies that $M' = \mathrm{ast(a, M)}$ and hence M' has a nonempty boundary, showing it is not homeomorphic to M. A small punctured neighborhood of a point in the boundary of M' has trivial homology while all small punctured neighborhoods of points in M has non vanishing $(d-1)$-th homology. This is what we mean by 'not even locally homologic': M and M' have homologically different sets of small punctured neighborhoods.

We are left to deal with the case $\mathrm{lk(a,M)} \subsetneqq \mathrm{L} \subsetneqq \mathrm{B(b)}$. As L is closed there exists a point $t \in L \cap \mathrm{int(B(b))}$ with a small punctured neighborhood $N(t, M')$ which is not contained in L. For a subspace K of M' denote by $N(t, K)$ the neighborhood in K $N(t, M') \cap K$. Thus $N(t, M') = N(t, \mathrm{ast(a,M)}) \cup_{\mathrm{N(t,L)}} \mathrm{N(t, B(b))}$. We get a Mayer-Vietoris exact sequence in reduced homology:

$$\begin{gathered} H_{d-1}N(t, L) \to H_{d-1}N(t, \mathrm{ast(a,M)}) \oplus \mathrm{H_{d-1}N(t, B(b))} \to \mathrm{H_{d-1}N(t, M')} \to \\ H_{d-2}N(t, L) \to H_{d-2}N(t, \mathrm{ast(a,M)}) \oplus \mathrm{H_{d-2}N(t, B(b))}. \end{gathered} \tag{5.9}$$

Note that $N(t, \mathrm{ast(a,M)})$ and $N(t, B(b))$ are homotopic to their boundaries which are $(d-1)$-spheres. Note further that $N(t, L)$ is homotopic to a proper subset X of $\partial(N(t, B(b)))$ such that the pair $(\partial(N(t, B(b))), X)$ is triangulated. By Alexander duality $H_{d-1}N(t, L) = 0$. Thus, (5.9) simplifies to the exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_{d-1}N(t, M') \to H_{d-2}N(t, L) \to 0.$$

Thus, $\mathrm{rank(H_{d-1}N(t, M'))} \geq 2$, hence M and M' are not locally homologic, and in particular are not homeomorphic. $\square$

Remarks: (1) Omitting the assumption in Theorem 5.4.1 that the boundary is empty makes both implications incorrect. Contracting an edge to a point shows that the Link Condition is not sufficient. Contracting an edge on the boundary of a cone over an empty triangle shows that the Link Condition is not necessary.

(2) The necessity of the Link Condition holds also in the topological category (and not only in the PL category), as the proof of Theorem 5.4.1 shows. Indeed, for this part we only used the fact that $B(b)$ is a pseudo manifold with boundary $\mathrm{lk(a,M)}$ (not that it is a ball); taking the point t to belong to exactly two facets of $B(b)$. For sufficiency of the Link Condition in the topological category, see Problem 3 in Section 5.5 below.

Walkup [75] mentioned, without details, the necessity of the Link Condition for contractions in topological manifolds, as well as the sufficiency of the Link Condition for the 3 dimensional case (where the category of PL-manifolds coincides with the topological one); see [75], p.82-83.

5.4.2 PL spheres

Definition 5.4.2 *Boundary complexes of simplices are* strongly edge decomposable *and, recursively, a triangulated PL-manifold S is* strongly edge decomposable *if it has an edge which satisfies the Link Condition (5.4) such that both its link and its contraction are strongly edge decomposable.*

By Theorem 5.4.1 the complexes in Definition 5.4.2 are all triangulated PL-spheres. Note that every 2-sphere is strongly edge decomposable.

Let vu be an edge in a simplicial complex K which satisfies the Link Condition, whose contraction $u \mapsto v$ results in the simplicial complex K'. Note that the f-polynomials satisfy

$$f(K,t) = f(K',t) + t(1+t)f(\mathrm{lk}(\{\mathrm{vu}\},\mathrm{K}),\mathrm{t}),$$

hence the h-polynomials satisfy

$$h(K,t) = h(K',t) + th(\mathrm{lk}(\{\mathrm{vu}\},\mathrm{K}),\mathrm{t}). \tag{5.10}$$

We conclude the following:

Corollary 5.4.3 *The g-vector of strongly edge decomposable triangulated spheres is non negative.* $\square$

Is it also an M-vector? Compare with Theorem 4.6.5. The strongly edge decomposable spheres (strictly) include the family of triangulated spheres which can be obtained from the boundary of a simplex by repeated Stellar subdivisions (at any face); the later are polytopal, hence their g-vector is an M-sequence. For the case of subdividing only at edges (5.10) was considered by Gal ([25], Proposition 2.4.3).

5.5 Open problems

1. Prove that if $H < K$ and K is embeddable in the m-sphere then H is embeddable in the m-sphere.

2. Let K be a triangulated $2d$-sphere and let T be a missing d-face in K. Let $L = K_d \cup \{T\}$. Show that L does not embed in $\mathbb{R}^{2d}$.

3. Given an edge in a triangulation of a compact manifold without boundary which satisfies the Link Condition, is it true that its contraction results in a homeomorphic space? Or at least in a space of the same homotopic or homological type?

   A Mayer-Vietoris argument shows that such topological manifolds M and M' have the same Betti numbers; both $\overline{\mathrm{st}}(a,M)$ and $B(b)$ are cones and hence their reduced homology vanishes.

   A candidate for a counterexample for Problem 3 may be the join $M = T * P$ where T is the boundary of a triangle and P a triangulation of Poincaré homology 3-sphere, where an edge with one vertex in T and the other in P satisfies the Link Condition. By the double-suspension theorem (Edwards [37] and Cannon [13]) M is a topological 5-sphere.

4. Show that the g-vector of strongly edge decomposable triangulated spheres is an M-vector.

Chapter 6

Face Rings for Graded Posets

6.1 Some classic f-vector results for graded posets

Let us review the characterization of f-vectors of finite simplicial complexes, known as the Schützenberger-Kruskal-Katona theorem (see [10] for a proof and for references). For any two integers $k, n > 0$ there exists a unique expansion

$$n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + ... + \binom{n_i}{i} \tag{6.1}$$

such that $n_k > n_{k-1} > ... > n_i \geq i \geq 1$ (details in [10]). Define the function ∂_{k-1} by

$$\partial_{k-1}(n) = \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + ... + \binom{n_i}{i-1}, \quad \partial_{k-1}(0) = 0.$$

Theorem 6.1.1 (Schützenberger-Kruskal-Katona) *f is the f-vector of some simplicial complex iff f ultimately vanishes and*

$$\forall k \geq 0 \quad 0 \leq \partial_k(f_k) \leq f_{k-1}. \tag{6.2}$$

For a ranked meet semi-lattice P, finite at every rank, let f_i be the number of elements with rank $i+1$ in P, and set $\text{rank}(\hat{0}) = 0$ where $\hat{0}$ is the minimum of P. The f-vector of P is $(f_{-1}, f_0, f_1, ...)$.

P has the *diamond property* if for every $x, y \in P$ such that $x < y$ and $\text{rank}(y) - \text{rank}(x) = 2$ there exist at least two elements in the open interval (x, y). The closed interval is denoted by $[x, y] = \{z \in P : x \leq z \leq y\}$.

We identify a simplicial complex with the poset of its faces ordered by inclusion. The following generalization of Theorem 6.1.1 is due to Wegner [77].

Theorem 6.1.2 (Wegner) *Let P be a finite ranked meet semi-lattice with the diamond property. Then its f-vector ultimately vanishes and satisfies (6.2).*

For $\hat{x} \in P$ define $P(\hat{x}) = \{x \in P : \hat{x} \leq x\}$ and let $y' \prec y$ denote y covers y'.

Lemma 6.1.3 *For a ranked meet semi-lattice P, the diamond property is equivalent to satisfying the following condition:*

(*) For every $\hat{x} \in P$, x which covers $\hat{x}$ and y such that $y \in P(\hat{x})$ and $y \neq \hat{x}$, there exists $y' \in P(\hat{x})$ such that $y' \prec y$ and $x \nleq y'$.

A multicomplex (on a finite ground set) can be considered as an order ideal of monomials I (i.e. if $m|n \in I$ then also $m \in I$) on a finite set of variables. Its f-vector is defined by $f_i = |\{m \in I : \deg(m) = i+1\}|$ (again $f_{-1} = 1$). Define the function ∂^{k-1} by

$$\partial^{k-1}(n) = \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + ... + \binom{n_i - 1}{i-1}, \quad \partial^{k-1}(0) = 0,$$

w.r.t the expansion (6.1).

Theorem 6.1.4 (Macaulay) *(simpler proofs in [15, 67]) f is the f-vector of some multicomplex iff $f_{-1} = 1$ and*

$$\forall k \geq 0 \quad 0 \leq \partial^k(f_k) \leq f_{k-1}. \tag{6.3}$$

Definition 6.1.5 *(Parallelogram property) A ranked poset P is said to have the* parallelogram property *if the following condition holds:*

(**) For every $\hat{x} \in P$ and $y \in P(\hat{x})$ such that $y \neq \hat{x}$, if the chain $\{\hat{x} = x_0 \prec x_1 \prec ... \prec x_r\}$ equals the closed interval $[\hat{x}, x_r]$ ($r > 0$) and is maximal w.r.t. inclusion such that $r < \mathrm{rank}(y)$ (the rank of y in the poset $P(\hat{x})$), and if $x_i < y$ and $x_{i+1} \nleq y$ for some $0 < i \leq r$, then there exists $y' \in P(\hat{x})$ such that $y' \prec y$, $x_{i-1} < y'$ and $x_i \nleq y'$. For $i = r$ interpret $x_{r+1} \nleq y$ as: $[\hat{x}, y]$ is not a chain.

See Figure 6.1 for an illustration of the parallelogram property. Note that condition (*) of Lemma 6.1.3 implies condition (**) of Definition 6.1.5 (with 1 being the only possible value of r). Posets of multicomplexes, polyhedral complexes, and rooted trees, satisfy the parallelogram property.
We identify a multicomplex with the poset of its monomials ordered by division. We now generalize Theorem 6.1.4; the proof is combinatorial.

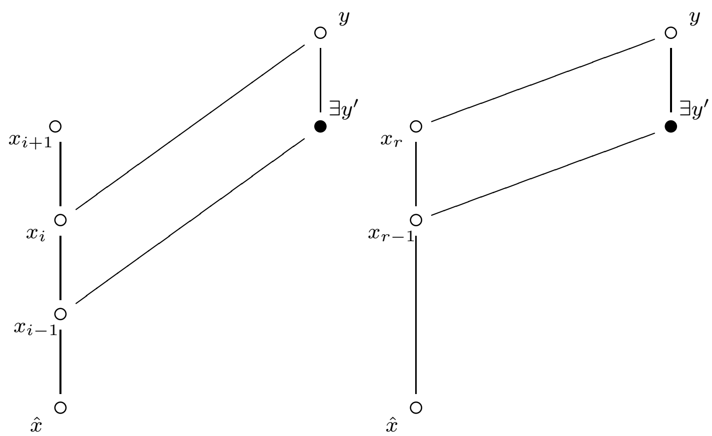

Figure 6.1: The parallelogram property for $i < r$ (left) and for $i = r$ (right).

Theorem 6.1.6 ([55], Theorem 1.6) *Let P be a ranked meet semi-lattice, finite at every rank, with the parallelogram property. Then its f-vector satisfies (6.3) and $f_{-1}(P) = 1$.*

For generalizations of Macaulay's theorem in a different direction ('compression'), see e.g. [15, 76].

In Section 1.2 we presented the symmetric and exterior face rings of a simplicial complex, to which we applied the shifting operator. The graded components of the face ring have dimensions corresponding to the f-vector. Shifting changes the bases of these components, hence preserves the f-vector, and results in a shifted complex, for which e.g. the inequalities $\partial_k(f_k) \leq f_{k-1}$ are easier to prove.

Analogous algebraic object and operator for more general graded posets are desirable in order to prove f-vector theorems for them. An algebraic object that will correspond to Macaulay inequalities, may help to settle the following well known conjecture:

Conjecture 6.1.7 *The toric g-vector of a (non-simplicial) polytope is an M-sequence, i.e. it satisfies Macaulay inequalities (6.3).*

Recently Karu [36] proved that the toric g-vector of a polytope is nonnegative, using a (complicated) graded module. Can we shift this structure, in order to prove Conjecture 6.1.7? A problem is that Karu's structure is a module over the polynomial ring with d (=dimension) variables, and not with n (=number of vertices) variables.

In the next two sections we make initial steps in this program.

6.2 Algebraic shifting for geometric meet semi-lattices

We will associate an analogue of the exterior face ring to geometric ranked meet semi-lattices, which coincides with the usual construction for the case of simplicial complexes. Applying algebraic shifting we construct a canonically defined shifted simplicial complex, having the same f-vector as its geometric meet semi-lattice.

Let $(L, <, r)$ be a ranked atomic meet semi-lattice with L the set of its elements, $<$ the partial order relation and $r : L \to \mathbb{N}$ its rank function. We denote it in short by L. L is called *geometric* if

$$r(x \wedge y) + r(x \vee y) \leq r(x) + r(y) \tag{6.4}$$

for every $x, y \in L$ such that $x \vee y$ exists. For example, the intersections of a finite collection of hyperplanes in a vector space form a geometric meet semi-lattice w.r.t. the reverse inclusion order and the codimension rank. Face posets of simplicial complexes are important examples of geometric meet semi-lattices, where (6.4) holds with equality.

Adding a maximum to a ranked meet semi-lattice makes it a lattice, denoted by $\hat{L}$, but the maximum may not have a rank. Denote by $\hat{0}, \hat{1}$ the minimum and maximum of $\hat{L}$, respectively, and by L_i the set of rank i elements in L. $r(\hat{0}) = 0$.

We now define the algebra $\bigwedge L$ over a field k with characteristic 2. Let V be a vector space over k with basis $\{e_u : u \in L_1\}$. Let $I_L = I_1 + I_2 + I_3$ be the ideal in the exterior algebra $\bigwedge V$ defined as follows. Choose a total ordering of L_1, and denote by e_S the wedge product $e_{s_1} \wedge ... \wedge e_{s_{|S|}}$ where $S = \{s_1 < ... < s_{|S|}\}$. Define:

$$I_1 = (e_S : S \subseteq L_1, \vee S = \hat{1} \in \hat{L}), \tag{6.5}$$

$$I_2 = (e_S : S \subseteq L_1, \vee S \in L, r(\vee S) \neq |S|), \tag{6.6}$$

$$I_3 = (e_S - e_T : T, S \subseteq L_1, \vee T = \vee S \in L, r(\vee S) = |S| = |T|, S \neq T). \tag{6.7}$$

(As $\mathrm{char}(k) = 2$, $e_S - e_T$ is independent of the ordering of the elements in S and in T.) Let $\bigwedge L = \bigwedge V / I_L$. As I_L is generated by homogeneous elements, $\bigwedge L$ inherits a grading from $\bigwedge V$. Let $f(\bigwedge L) = (f_{-1}, f_0, ..)$ be its graded dimensions vector, i.e. f_{i-1} is the dimension of the degree i component of $\bigwedge L$.

Remark: If L is the poset of a simplicial complex, then $I_L = I_1$ and $\bigwedge L$ is the classic exterior face ring of L, as in [30].

The following proposition will be used for showing that $\bigwedge L$ and L have the same f-vector. Its easy proof by induction on the rank is omitted.

Proposition 6.2.1 *Let L be a geometric ranked meet semi-lattice. Let $l \in L$ and let S be a minimal set of atoms such that $\vee S = l$, i.e. if $T \subsetneq S$ then $\vee T < l$. Then $r(l) = |S|$.* $\square$

Remark: The converse of Proposition 6.2.1 is also true: Let L be a ranked atomic meet semi-lattice such that every $l \in L$ and every minimal set of atoms S such that $\vee S = l$ satisfy $r(l) = |S|$. Then L is geometric.

Proposition 6.2.2 $f(\bigwedge L) = f(L)$.

Proof: Denote by $\tilde{w}$ the projection of $w \in \bigwedge V$ on $\bigwedge L$. We will show that picking $S(l)$ such that $S(l) \subseteq L_1, \vee S(l) = l, |S(l)| = r(l)$ for each $l \in L$ gives a basis over k of $\bigwedge L$, $E = \{\tilde{e}_{S(l)} : l \in L\}$.

As $\{\tilde{e}_S : S \subseteq L_1\}$ is a basis of $\bigwedge V$, it is clear from the definition of I_L that E spans $\bigwedge L$. To show that E is independent, we will prove first that the generators of I_L as an ideal, that are specified in (6.6), (6.5) and (6.7), actually span it as a vector space over k.

As $x \vee \hat{1} = \hat{1}$ for all $x \in L$, the generators of I_1 that are specified in (6.5) span it as a k-vector space. Next, we show that the generators of I_2 and I_1 that are specified in (6.6) and in (6.5) respectively, span $I_1 + I_2$ as a k-vector space: if e_S is such a generator of I_2 and $U \subseteq L_1$ then either $e_U \wedge e_S \in I_1$ (if $U \cap S \neq \emptyset$ or if $\vee(U \cup S) = \hat{1}$) or else, by Proposition 6.2.1, $r(\vee(U \cup S)) < |U \cup S|$ and hence $e_U \wedge e_S$ is also such a generator of I_2.

Let $e_S - e_T$ be a generator of I_3 as specified in (6.7) and let $U \subseteq L_1$. If $U \cap T \neq \emptyset$ then $e_T \wedge e_U = 0$ and $e_S \wedge e_U$ is either zero (if $U \cap S \neq \emptyset$) or else a generator of $I_1 + I_2$, by Proposition 6.2.1; and similarly when $U \cap S \neq \emptyset$. If $U \cap T = \emptyset = U \cap S$ then $\vee(S \cup U) = \vee(T \cup U)$ and $|S \cup U| = |T \cup U|$. Hence, if $e_S \wedge e_U - e_T \wedge e_U$ is not the obvious difference of two generators of I_1 or of I_2 as specified in (6.5) and (6.6), then it is a generator of I_3 as specified in (6.7). We conclude that these generators of I_L as an ideal span it as a vector space over k.

Assume that $\sum_{l \in L} a_l \tilde{e}_{S(l)} = 0$, i.e. $\sum_{l \in L} a_l e_{S(l)} \in I_L$ where $a_l \in k$ for all $l \in L$. By the discussion above, $\sum_{l \in L} a_l e_{S(l)}$ is in the span (over k) of the generators of I_3 that are specified in (6.7). But for every $l \in L$ and every such generator g of I_3, if $g = \sum\{b_S e_S : \vee S \in L, r(\vee S) = |S|\}$ ($b_S \in k$ for all S) then $\sum\{b_S : \vee S = l\} = 0$. Hence $a_l = 0$ for every $l \in L$. Thus E is a basis of $\bigwedge L$, hence $f(\bigwedge L) = f(L)$. $\square$

Now let us shift. Note that exterior algebraic shifting, which was defined for the exterior face ring, can be applied to any graded exterior algebra finitely generated by degree 1 elements. It results in a simplicial complex with an f-vector that is equal to the vector of graded dimensions of the

algebra. This shows that any such graded algebra satisfies Kruskal-Katona inequalities! We apply this construction to $\bigwedge L$:

Let $B = \{b_u : u \in L_1\}$ be a basis of V. Then $\{\tilde{b}_S : S \subseteq L_1\}$ spans $\bigwedge L$. Choosing a basis from this set in the greedy way w.r.t. the lexicographic order $<_L$ on equal sized sets, defines a collection of sets:

$$\Delta_B(L) = \{S : \tilde{b}_S \notin \mathrm{span}_k\{\tilde{b}_T : |T| = |S|, T <_L S\}\}.$$

$\Delta_B(L)$ is a simplicial complex, and by Proposition 6.2.2 $f(\Delta_B(L)) = f(L)$. For a generic B, $\Delta_B(L)$ is shifted. Moreover, the construction is canonical, i.e. is independent both of the chosen ordering of L_1 and of the generically chosen basis B. It is also independent of the characteristic 2 field that we picked. We denote $\Delta(L) = \Delta_B(L)$ for a generic B. For proofs of the above statements we refer to Björner and Kalai [8] (they proved for the case where L is a simplicial complex, but the proofs remain valid for any graded exterior algebra finitely generated by degree 1 elements).

We summarize the above discussion in the following theorem:

Theorem 6.2.3 *Let L be a geometric meet semi-lattice, and let k be a field of characteristic 2. There exists a canonically defined shifted simplicial complex $\Delta(L)$ associated with L, with $f(\Delta(L)) = f(L)$.* $\square$

Remarks: (1) The fact that L satisfies Kruskal-Katona inequalities follows also without using our algebraic construction, from the fact that it satisfies the diamond property and applying Theorem 6.1.2. The diamond property is easily seen to hold for all ranked atomic meet semi-lattices.

(2) A different operation, which does depend on the ordering of L_1 and results in a simplicial complex with the same f-vector, was described by Björner [7], Chapter 7, Problem 7.25: totally order L_1. For each $x \in L$ choose the lexicographically least subset $S_x \subseteq L_1$ such that $\vee S_x = x$ ($S_{\hat{0}} = \emptyset$). Define $\Delta_<(L) = \{S_x : x \in L\}$. Then $\Delta_<(L)$ is a simplicial complex with the same f-vector as L. An advantage in our operation is that it is canonical (and results in a shifted simplicial complex). To see that these two operations are indeed different, let L be the face poset of a simplicial complex. Then for any total ordering of L_1, $\Delta_<(L) = L$. But if the simplicial complex is not shifted (e.g. a 4-cycle), then $\Delta(L) \neq L$.

6.3 Algebraic shifting for generalized multicomplexes

We will associate an analogue of the symmetric (Stanley-Reisner) face ring with a common generalization of multicomplexes and geometric meet semi-

lattices. Applying an algebraic shifting operation, we construct a multicomplex having the same f-vector as the original poset.

Let $\mathbb{P}$ be the following family of posets: to construct $P \in \mathbb{P}$ start with a geometric meet semi-lattice L. Associate with each $l \in L$ the (square free) monomial $m(l) = \prod_{a<l, a\in L_1} x_a$, and equip it with rank $r(m(l)) = r(l)$. Denote this collection of monomials by M_0. Now repeat the following procedure finitely or countably many times to construct ($M_0 \subseteq M_1 \subseteq ...$): Choose $m \in M_i$ and $a \in L$ such that $x_a|m$, $\frac{x_a}{x_b}m \in M_i$ for all $b \in L_1$ such that $x_b|m$, and $x_a m \notin M_i$. M_{i+1} is obtained from M_i by adding $x_a m$, setting its rank to be $r(x_a m) = r(m) + 1$ and let it cover all the elements $\frac{x_a}{x_b}m$ where $b \in L_1$ such that $x_b|m$. Define $P = \cup M_i$.

Note that the posets in $\mathbb{P}$ are ranked (not necessarily atomic) meet semi-lattices with the parallelogram property, and that $\mathbb{P}$ includes all multicomplexes (start with L, a simplicial complex) and geometric meet semi-lattices ($P = M_0$).

For $P \in \mathbb{P}$ define the following analogue of the Stanley-Reisner ring: Assume for a moment that P is finite. Fix a field k, and denote $P_1 = \{1, .., n\}$. Let $A = k[x_1, .., x_n]$ be a polynomial ring. For j such that $1 \leq j \leq n$ let r_j be the minimal integer number such that $x_j^{r_j+1}$ does not divide any of the monomials $p \in P$. Note that each $i \in P$ of rank 1 belongs to a unique maximal interval which is a chain; whose top element is $x_i^{r_i}$. By abuse of notation, we identify the elements in such intervals with their corresponding monomials in A.

We add a maximum $\hat{1}$ to P to obtain $\hat{P}$ and define the following ideals in A:

$I_0 = (\prod_{i=1}^n x_i^{a_i} : \exists j \ 1 \leq j \leq n, \ a_j > r_j)$,

$I_1 = (\prod_{i=1}^n x_i^{a_i} : \forall j \ a_j \leq r_j, \ \vee_{i=1}^n x_i^{a_i} = \hat{1} \in \hat{P})$,

$I_2 = (\prod_{i=1}^n x_i^{a_i} : \vee_{i=1}^n x_i^{a_i} \in P, r(\vee_{i=1}^n x_i^{a_i}) \neq \sum_i a_i)$,

$I_3 = (\prod_{i=1}^n x_i^{a_i} - \prod_{i=1}^n x_i^{b_i} : \vee_{i=1}^n x_i^{a_i} = \vee_{i=1}^n x_i^{b_i} \in P, r(\vee_{i=1}^n x_i^{a_i}) = \sum_i a_i = \sum_i b_i)$,

$I_P = I_0 + I_1 + I_2 + I_3$.

Define $k[P] := A/I_P$. As I_P is homogeneous, $k[P]$ inherits a grading from A. Let $f(k[P]) = (f_{-1}, f_0, ..)$ where $f_i = \dim_k\{m \in k[P] : r(m) = i + 1\}$ ($f_{-1} = 1$).

The proof of the following proposition is similar to the proof of Proposition 6.2.2, and is omitted.

Proposition 6.3.1 $f(k[P]) = f(P)$. $\square$

Denote by $\tilde{w}$ the projection of $w \in A$ on $k[P]$. Let $B = \{y_1, .., y_n\}$ be a basis

of A_1. Then

$$\Delta_B(P) := \{\prod_{i=1}^{n} y_i^{a_i} : \prod_{i=1}^{n} \tilde{y_i}^{a_i} \notin \mathrm{span}_k\{\prod_{i=1}^{n} \tilde{y_i}^{b_i} : \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i, \prod_{i=1}^{n} y_i^{b_i} <_L \prod_{i=1}^{n} y_i^{a_i}\}\}$$

is an order ideal of monomials with an f-vector $f(P)$. (The lexicographic order on monomials of equal degree is defined by $\prod_{i=1}^{n} y_i^{b_i} <_L \prod_{i=1}^{n} y_i^{a_i}$ iff there exists j such that for all $1 \leq t < j$ $a_t = b_t$ and $b_j > a_j$.) To prove this, we reproduce the argument of Stanley for proving Macaulay's theorem ([68], Theorem 2.1): as the projections of the elements in $\Delta_B(P)$ form a k-basis of $k[P]$, then by Proposition 6.3.1 $f(\Delta_B(P)) = f(P)$. If $m \notin \Delta_B(P)$ then $m = \sum\{a_n n : \deg(n) = \deg(m), n <_L m\}$, hence for any monomial m' $m'm = \sum\{a_n m'n : \deg(n) = \deg(m), n <_L m\}$. But $\deg(m'm) = \deg(m'n)$ and $m'n <_L m'm$ for these n's, hence $m'm \notin \Delta_B(P)$, thus $\Delta_B(P)$ is an order ideal of monomials.

Remark: For B a generic basis the construction is canonical in the same sense as defined for the exterior case.

Combining Proposition 6.3.1 with Theorem 6.1.4 we obtain

Corollary 6.3.2 *Every $P \in \mathbb{P}$ satisfies Macaulay inequalities (6.3).* $\square$

Remark: If P is infinite, let $P_{\leq r} := \{p \in P : r(p) \leq r\}$ and construct $\Delta(P_{\leq r})$ for each r. Then $\Delta(P_{\leq r}) \subseteq \Delta(P_{\leq r+1})$ for every r, and $\Delta(P) := \cup_r \Delta(P_{\leq r})$ is an order ideal of monomials with f-vector $f(P)$. Hence, Corollary 6.3.2 holds in this case too.

6.4 Open problems

1. The Kruskal-Katona inequalities hold for any meet semi lattice with the diamond property (Theorem 6.1.2). Can an algebraic proof be given?

2. Similarly, can Theorem 6.1.6 be proved algebraically?

3. Prove Conjecture 6.1.7 by applying shifting to a suitable algebraic object.

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